Chapter 11: Probability of Compound Events

1. (11.1) Compound Events
2. (11.2) Probability of a Compound Event
3. (11.3) Probability Viewed as Darts Tossed at a Dartboard

Motivating Example: Blood Type & Rh-Classification

- One example of a compound event is blood typing together with the Rh-classification of blood
- Given the blood types *and* Rh-classifications of an offspring's parents, we could ask:
  - What is the probability a child is Rh-positive **and** type B?
  - What is the probability a child is Rh-positive **or** type B?
  - What is the probability a child is **neither** Rh-positive **nor** type B?
- Recall that 'Rh-positive' and 'type B' are events (subsets) of the sample space (made up of all elementary events) for the experiment that records the blood type and Rh-classification for an offspring of parents with known blood types and Rh-classifications
- A **compound event** is an event that is some logical combination of events

Types of Compound Events

Consider an experiment with sample space $S = \{e_1, e_2, \ldots, e_n\}$. Let E and F be two events in S (subsets of S), and let e be a particular outcome (elementary event) of the experiment. Here are the various types of compound events:

- **E or F**
  - The event "E or F" (denoted $E \cup F$) has occurred if e is either in event E or in event F or in both E and F.

Example 11.1 (Blood Typing: E ∪ F)

Suppose our sample space is blood types with Rh-factor:

$$S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}$$

Let E be the event "type B blood" and let F be the event "Rh-positive." Identify the events E, F, and E or F (E ∪ F).

Solution:

$$E = \{B+, B-\}$$

$$F = \{A+, B+, AB+, O+\}$$

$$E \text{ or } F = E \cup F = \{A+, B+, B-, AB+, O+\}$$

Types of Compound Events

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- **E or F**
  - The event "E or F" (denoted $E \cup F$) has occurred if e is either in event E or in event F or in both E and F.
- **E and F**
  - The event "E and F" (denoted $E \cap F$) has occurred if e is in event E and in event F.

Example 11.2 (Blood Typing: E∩F)

Suppose our sample space is blood types with Rh-factor:

$$S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}$$

Let E be the event "type B blood" and let F be the event "Rh-positive." Identify the events E, F, and E and F (E∩F).

Solution:

$$E = \{B+, B-\}$$

$$F = \{A+, B+, AB+, O+\}$$

$$E \text{ and } F = E \cap F = \{B+\}$$

Types of Compound Events

Consider an experiment with sample space $S = \{e_1, e_2, \dots, e_n\}$. Let E and F be two events in S (subsets of S), and let e be a particular outcome (elementary event) of the experiment. Here are the various types of compound events:

- **E or F**
  - The event "E or F" (denoted $E \cup F$) has occurred if e is either in event E or in event F or in both E and F.
- **E and F**
  - The event "E and F" (denoted $E \cap F$) has occurred if e is in event E and in event F.
- **Not E**
  - The event "not E" (denoted $\bar{E}$ or $\neg E$ and called the **complement** of E) has occurred if e is not in event E.

Example 11.3 (Blood Typing: Ē)

Suppose our sample space is blood types with Rh-factor:

$$S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}$$

Let E be the event "type B blood." Identify the events E and not E (Ē).

Solution:

$$E = \{B+, B-\}$$

$$not\ E = \overline{E} = \neg E = \{A+, A-, AB+, AB-, O+, O-\}$$

Types of Compound Events

Consider an experiment with sample space $S = \{e_1, e_2, \ldots, e_n\}$. Let E and F be two events in S (subsets of S), and let e be a particular outcome (elementary event) of the experiment. Here are the various types of compound events:

- **E or F**
  - The event "E or F" (denoted $E \cup F$) has occurred if e is either in event E or in event F or in both E and F.
- **E and F**
  - The event "E and F" (denoted $E \cap F$) has occurred if e is in event E and in event F.
- **Not E**
  - The event "not E" (denoted $\bar{E}$ or $\neg E$ and called the **complement** of E) has occurred if e is not in event E.
- **E but not F**
  - The event "E but not F" (denoted $E \cap \bar{F}$ or $E \cap \neg F$ or E-F) has occurred if e is in event E and not in event F.

Example 11.4 (Blood Typing: $E \cap \overline{F}$)

Suppose our sample space is blood types with Rh-factor:

$$S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}$$

Let E be the event "type B blood" and let F be the event "Rh-positive." Identify the events E, F, and E but not F ($E \cap \overline{F}$).

Solution:

$$E = \{B+, B-\}$$

$$F = \{A+, B+, AB+, O+\}$$

$$E \text{ but not } F = E \cap \overline{F} = \{B-\}$$

1. (11.1) Compound Events

Mutually Exclusive Events

- There is a special event that all experiments have in common- the impossible event. As a set (events are sets), the impossible event is called the **empty set** or **null set** and is denoted $\varnothing$ or { }.
- Events E and F are said to be **mutually exclusive** if their intersection is empty; that is, if $E \cap F = \varnothing$
- In the next section we will learn that if E and F are mutually exclusive, then the probability of $E \cup F$ is the sum of the probabilities; that is, $P(E \cup F) = P(E) + P(F)$
- We note that, in particular, elementary events are mutually exclusive. This is why we were able to add the probabilities of the elementary events as we did in Example 11.8.

Example 11.5 (Blood Typing: E∩F = {∅})

Suppose our sample space is blood types with Rh-factor:

$$S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}$$

Let E be the event "type B blood" and let F be the event "type O blood." Identify the events E, F, and E and F (E∩F).

Solution:

$$E = \{B+, B-\}$$

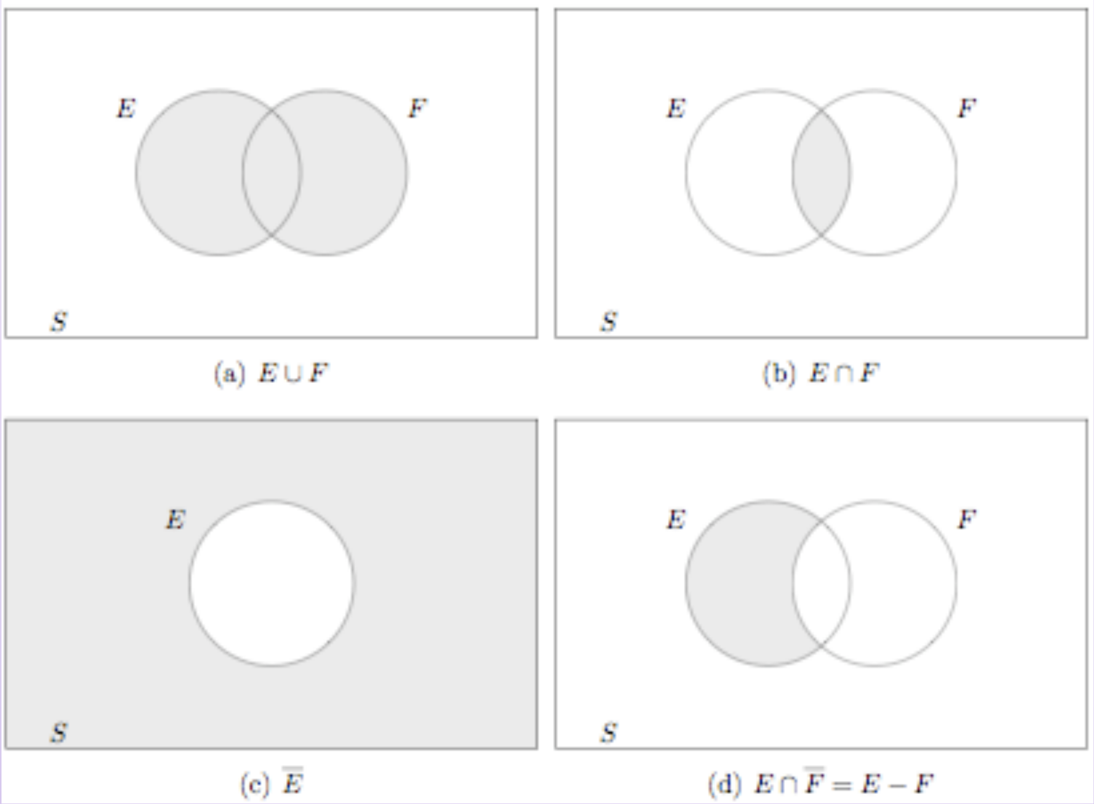
$$F = \{O+, O-\}$$

$$E \cap F = \varnothing$$

Thus, events E and F are mutually exclusive.

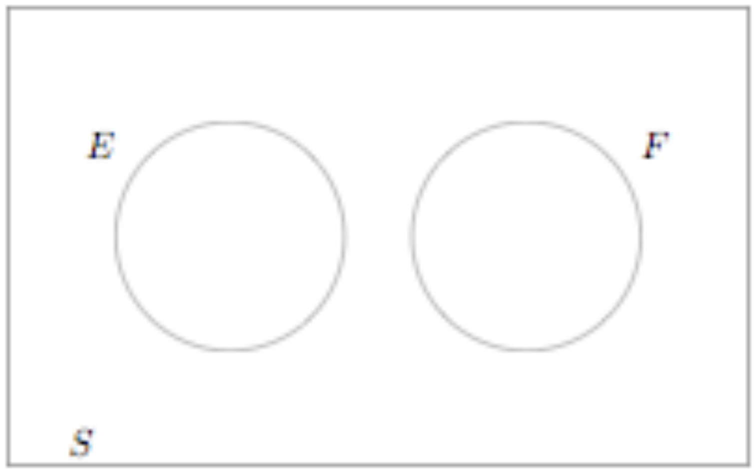
Venn Diagrams: Compound Events

It is convenient to summarize the previous discussion using Venn diagrams. The four compound events:



Venn Diagrams: Mutually Exclusive Events

As sets, mutually exclusive events are sets that do not intersect:



(e) $E \cap F = \emptyset$

Probability Axioms

Let S be a sample space. Then:

1. $P(S) = 1$
2. If E is an event in S, then $0 \leq P(E) \leq 1$
3. If A and B are mutually exclusive events in S (i.e. $A \cap B = \emptyset$), then:
   a) $P(A \cup B) = P(A) + P(B)$
   b) $P(A \cap B) = 0$

From these simple, intuitive axioms we can derive several laws of probability.

Probability Axioms: Example 11.6 (Eye Color)

Suppose we know the following about the probability of an individual's eye color:

$$P(\text{brown eyes}) = \frac{1}{2}, \quad P(\text{blue eyes}) = \frac{1}{8}, \quad P(\text{green eyes}) = \frac{3}{8}$$

What is the probability the individual has blue or green eyes?

Solution: Since the individual can only have one of these three eye colors, each of the above events is mutually exclusive. Thus, $P(\text{blue or green}) = P(\text{blue} \cup \text{green})$

$$= P(\text{blue}) + P(\text{green})$$

$$= \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

Probability Laws: P(Ē)

Let S be a sample space and let E be an event in S. Then:

$$P\left(\overline{E}\right) = 1 - P\left(E\right)$$

Let's see how this law comes from the axioms:

$$1 = P\left(S\right) = P\left(E \cup \overline{E}\right) = P\left(E\right) + P\left(\overline{E}\right)$$

$$\Rightarrow \qquad P\left(\overline{E}\right) = 1 - P\left(E\right)$$

Similarly, if A is an event, we have:

$$P\left(A\right) = 1 - P\left(\overline{A}\right)$$

Probability Laws: Example 11.7 (At Least One Girl)

A family has 3 children. What is the probability that there is at least one girl?

Solution: If the family has three children, and each child can be a boy or a girl, there are a total of 2 × 2 × 2 = 8 possible ways this family could be formed. Let event A = "at least one girl". We could list out all the ways there could be at least one girl among three children, or we could simply look at the complement of A. That is $\bar{A}$ = "all three children are boys". There is only one way to have all three boys, thus $P(\bar{A}) = 1/8$ and we have:

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{8} = \frac{7}{8}$$

Probability Laws: P(E-F)

Let S be a sample space and let E and F be events in S. Then:

$$P(E-F)=P(E\cap\overline{F})=P(E)-P(E\cap F)$$

Let's see how this law comes from the axioms:

Since $E=(E\cap F)\cup(E\cap\overline{F})$ and since these are mutually exclusive, we have:

$$P(E)=P(E\cap F)+P(E\cap\overline{F})$$

$$\Rightarrow \qquad P(E\cap\overline{F})=P(E)-P(E\cap F)$$

Probability Laws: P(E ∪ F)

Let S be a sample space and let E and F be events in S. Then:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Notice that if $E \cap F = \{\ \}$, then this law reduces to the axiom regarding the probability of the union of mutually exclusive events.

Let's see how this law comes from the axioms:

Since $E \cup F = (F) \cup (E \cap \overline{F})$ and since these are mutually exclusive, we have:

$$\begin{aligned} P(E \cup F) &= P(F) + P(E \cap \overline{F}) \\ &= P(F) + P(E) - P(E \cap F) \end{aligned}$$

Probability Laws: Example 11.9 (Rolling Dice)

In rolling two dice, what is the probability of rolling a sum of 6 or doubles?

Solution: Let events A = “sum of 6” and B = “doubles”. If we want to find $P(A \cup B)$ then we need to find P(A), P(B), and $P(A \cap B)$. In rolling two dice there are 6×6 = 36 possible outcomes. The elementary events in each event are:

$$A = \{15, 24, 33, 42, 51\}$$

$$B = \{11, 22, 33, 44, 55, 66\}$$

$$A \cap B = \{33\}$$

Thus: $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{5}{18}$$

Probability Laws: P(E ∪ F ∪ G)

Let S be a sample space and let E, F and G be events in S. Then:

$$
\begin{aligned}
P(E \cup F \cup G) = P(E) + P(F) + P(G) \\
-\left[P(E \cap F) + P(E \cap G) + P(F \cap G)\right] \\
+ P(E \cap F \cap G)
\end{aligned}
$$

Probability Laws: Example 11.11 (Beetles)

Suppose we have a population of beetles in which 30% have wings and the rest are wingless. You select a beetle at random and record whether or not it has wings, then put it back. You do this three times. What is the probability that on at least one sample you got a winged beetle?

Solution: You might think “the chance on each selection is .3, thus over three tries the chance of getting at least one beetle is 0.3 + 0.3 + 0.3 = 0.9.

But what if we had 4 selections? This reasoning would give us a probability of 1.2. This is not good because by our second probability axiom, any event can only have a probability between 0 and 1.

What is wrong with this thinking? “At least one winged beetle” means selecting a beetle on one of the 3 tries, or 2 of the 3 tries or on all 3 tries.

Probability Laws: Example 11.11 (Beetles)

Let's use a probability law. Let A="winged beetle on try 1", B="winged beetle on try 2", C ="winged beetle on try 3". Then we have:

$$
\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\
&\quad - \left[P(A \cap B) + P(A \cap C) + P(B \cap C)\right] \\
&\quad + P(A \cap B \cap C) \\
&= 0.3 + 0.3 + 0.3 - \left(0.3^2 + 0.3^2 + 0.3^2\right) + 0.3^3 \\
&\approx 0.657
\end{aligned}
$$

There's a much easier way to think about this problem. See the textbook for details- we will revisit this example in the next chapter.

DeMorgan's Laws

The following, known as DeMorgan's Laws, will greatly simplify the derivation of the last probability law. Notice that these are not probability laws.

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Probability Laws: P(Ē∩¬F)

Let S be a sample space and let E and F be events in S. Then:

$$P\left(\overline{E}\cap\overline{F}\right)=1-P\left(E\cup F\right)$$

Let's see how this law comes from the axioms:

Since $\overline{E}\cap\overline{F}=\overline{E\cup F}$ we have:

$$P\left(\overline{E}\cap\overline{F}\right)=P\left(\overline{E\cup F}\right)$$

$$=1-P\left(E\cup F\right)$$

See Example 11.10 in the book.

Probability Viewed as Darts Tossed at a Dartboard

Consider the experiment of throwing a dart at a pixelated dartboard:

- For each throw, we assume that the dart does, indeed, hit the board and it's location on the board is otherwise random.
- The sample space for this experiment is the set of all possible pixels.
- Consider the event, "Get a bullseye." This event is the set of pixels located in the demarcated region at the center of the board.
- Under this setup, how could we define the probability of getting a bullseye?

Probability Viewed as Darts Tossed at a Dartboard

- The probability of getting a bullseye is simply:

$$P(\text{bullseye}) = \frac{\#\ \text{pixels in bullseye region}}{\#\ \text{pixels (total)}}$$

- Now consider the same experiment but with a standard dartboard.
- The (idealized) sample space now has many more elements- instead of being restricted to a certain number of pixels, the dart can now land "anywhere." In this idealized sense, we can't even write down the sample space because there are infinitely many places on the board for the dart to land.
- Under this setup, how could we define the probability of getting a bullseye?

Probability Viewed as Darts Tossed at a Dartboard

- There is a very natural way to describe the probability. Rather than counting the number of pixels in the bullseye region and dividing by the total number of pixels, we calculate areas. That is,

$$P(\text{bullseye}) = \frac{\text{area of the bullseye region}}{\text{area (total) of the board}}$$

- This point of view is often helpful when thinking about probabilities of events. We will, in fact, adopt this point of view in Chapter 13 when we consider conditional probability.

Homework

Chapter 11: 1, 3, 4, 5, 6, 7, 8, 10, 11

Some answers:

11.1 a. {1, 2, 3, 5}, {1,3} b. {2, 4, 6} d. {4, 6} e. no

11.3 a. males who developed emphysema and are not heavy smokers b. g c. h

11.4 a. doubles or sum of 10, {(5,5), (1,1), (2,2), (3,3), (4,4), (6,6), (6,4), (4,6)} b. doubles and sum of 10 {(5,5)} c. a sum of 10 with no fives {(6, 4), (4,6)}

11.6 0.5

11.7 0.1, 0.3

11.8 a, d mutually exclusive

11.11 a. humans infected and are resistant to both drugs, 15%, b. 55% b U c