Chapter 12: Conditional Probability & Independence

1. (12.1) Conditional Probability
2. (12.2) Independence

Motivating Example

- In this chapter we explore finding the probability of an event *given* certain conditions or prior information.

- For example, consider the experiment of rolling two dice. The probability of the event A = "sum of 6" is easy to find:

$$P(A) = \frac{|A|}{|S|} = \frac{\left|\{15, 24, 33, 42, 51\}\right|}{|S|} = \frac{5}{36}$$

- Suppose, however, that you roll two dice and you don't look at them. I tell you that you rolled doubles. Now what is the probability that the sum is 6?

- Solution: Let event A = "sum of 6" and B = "rolled doubles." We now want to know the probability of event A *given* event B. Intuitively, we expect that the answer should be 1/6 since there are six ways to roll doubles and only one of them sums to 6.
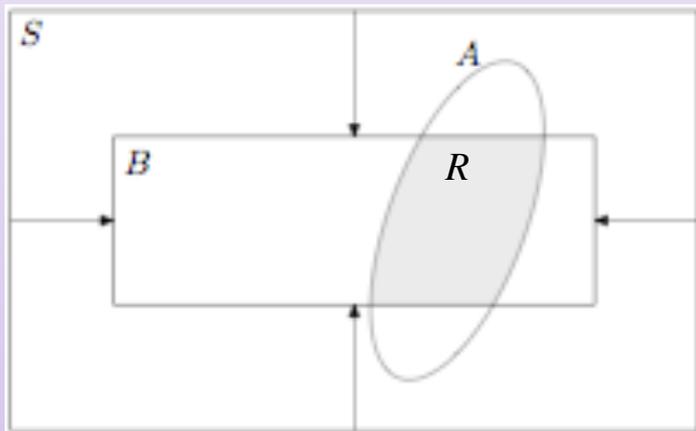
Definition: Conditional Probability

- Let B be an event with P(B) > 0. Then we define the **conditional probability** of event A given B by:

$$P\left(A|B\right) = \frac{P\left(A \cap B\right)}{P(B)} = \frac{(\text{area of R})/(\text{area of S})}{(\text{area of B})/(\text{area of S})} = \frac{\text{area of R}}{\text{area of B}}$$

- Heuristically, consider the diagram below and adopt the "dartboard" point of view described in Chapter 12:

Given that event B has occurred, we know that our "dart" has landed somewhere in B. Thus, the probability that it also landed in A is the area of the shaded region, R, divided by the area of rectangle B.

Motivating Example (Again)

- Recall our solution for the motivating example: Let event A = "sum of 6" and B = "rolled doubles." Intuitively, we expect that the the probability of A *given* B should be 1/6 since there are six ways to roll doubles and only one of them sums to 6.

- We now check this against our definition:

$$P(\text{A}|\text{B}) = \frac{P(A \cap B)}{P(B)} = \frac{\left|\{33\}\right|/|S|}{\left|\{11,22,33,44,55,66\}\right|/|S|} = \frac{1/36}{6/36} = \frac{1}{6}$$

Example 12.1 (Tay-Sachs Disease)

- Tay-Sachs disease is a serious disorder of the nervous system that usually results in death by age 2 or 3. Affected individuals have genotype tt, while normal (non-affected) individuals have genotype Tt or TT.

- Judy has a little brother with Tay-Sachs disease and is worried she may carry the recessive allele. What is the probability of this?

- Solution: First, notice that both of Judy's parents must be Tt if her little brother has the disease. (We assume that neither parent can be tt since they each lived to adulthood.)

Example 12.1 (Tay-Sachs Disease)

- The Punnett square gives the possible genotypes for Judy without being given any other information:

Since Judy does not have the disease, we eliminate the tt as a possibility. Looking at the Punnett square, we intuitively guess that the probability should be 2/3 . Let us show this is true using the definition of conditional probability.

- Let A = "Judy is Tt" and B = "Judy is not tt. Then,

$$P\left(\mathrm{A}\middle|\mathrm{B}\right) = \frac{P\left(A \cap B\right)}{P\left(B\right)} = \frac{\left|\left\{Tt,tT\right\}\right|/\left|S\right|}{\left|\left\{TT,Tt,tT\right\}\right|/\left|S\right|} = \frac{2/4}{3/4} = \frac{2}{3}$$

Example 12.2 (Drug Testing)

- A test for a new sleeping pill involved 200 individuals where 100 of the individuals were given the sleeping pill, and the other 100 were given a sugar pill. The results of the test are shown in the following table:

	Improved sleep	Did not sleep better
Sleeping pill	71	29
Sugar pill	58	42

What is the probability that if you take the sleeping pill you will sleep better?

- Solution: Let event A = "took sleeping pill," and event B = "slept better." We want to find the probability that you will sleep better given that you took the sleeping pill, i.e. P(B|A).

$$P(\text{B}|\text{A}) = \frac{P(B \cap A)}{P(A)} = \frac{|B \cap A|/|S|}{|A|/|S|} = \frac{71/200}{100/200} = 0.71$$

Example 12.2 (Drug Testing)

	Improved sleep	Did not sleep better
Sleeping pill	71	29
Sugar pill	58	42

What is the probability that if you slept better, you took the sleeping pill?

- Solution: (This is the converse of the previous question.) Let event A = "took sleeping pill," and event B = "slept better." Now we want to find P(A|B):

$$P\left(\mathrm{A}|\mathrm{B}\right) = \frac{P\left(A \cap B\right)}{P\left(B\right)} = \frac{\left|A \cap B\right|/\left|S\right|}{\left|B\right|/\left|S\right|} = \frac{71/200}{129/200} = 0.55$$

Some More Probability Laws: P(Ā|B)

- Let A & B be events in S. Then,

$$P\left(\overline{A}\middle|B\right) = 1 - P\left(A\middle|B\right)$$

- The derivation is straightforward:

$$P\left(\overline{A}\middle|B\right) = \frac{P\left(\overline{A} \cap B\right)}{P(B)} = \frac{P\left(B \cap \overline{A}\right)}{P(B)}$$

$$= \frac{P(B) - P(B \cap A)}{P(B)}$$

$$= 1 - \frac{P(A \cap B)}{P(B)} = 1 - P\left(A\middle|B\right)$$

Note, however, that this does **not** work for the other position. That is:

$$P\left(A\middle|\overline{B}\right) \neq 1 - P\left(A\middle|B\right)$$

Probability Law: P(A∩B)

- Let A & B be events in S. If If P(B) ≠ 0 then,

$$P\left(A \cap B\right) = P\left(A|B\right)P\left(B\right)$$

- The derivation follows immediately from the definition of conditional probability:

$$P\left(A|B\right) = \frac{P\left(A \cap B\right)}{P\left(B\right)} \Rightarrow P\left(A \cap B\right) = P\left(A|B\right)P\left(B\right)$$

- Notice that since set intersection is commutative (that is, A∩B = B∩A), we automatically get:

$$P\left(A \cap B\right) = P\left(B|A\right)P\left(A\right)$$

Example 12.3.a (Beetle Sampling without Replacement)

We have a population of 150 beetles. Thirty percent have wings and the rest are wingless. You select a beetle at random and record whether or not it has wings, and do not put it back. Then you select another beetle.

What is the probability that the first and second beetle have wings?

Solution: Let event A = "first beetle is winged," and event B = "second beetle is winged." First we note that, at the start, there are 0.3×150 = 45 winged beetles. Then:

$$P(B \cap A) = P(B|A)P(A)$$

$$= \left(\frac{44}{149}\right) \cdot \left(\frac{45}{150}\right) \approx 0.0886$$

Definition

Suppose the probability of some event is not affected by the occurrence of another event. It seems natural in such a case to consider these events independent of one another. We formalize the notion:

We say events A and B are **independent** if P(A|B) = P(A).

Notice that, formally, if we swap the letters A & B in the definition, we get P(B|A) = P(B). We should verify that this is consistent with our definition for conditional probability: Suppose that A and B are independent. Then,

$$P(A) = P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow \boxed{P(A)P(B) = P(A \cap B)}$$

We want to take special note of this relationship.

$$\Rightarrow P(B) = \frac{P(B \cap A)}{P(A)} = P(B|A)$$

Probability Law

- Let A & B be *independent* events in S. Then,

$$P(A \cap B) = P(A)P(B)$$

- Notice, when two events are mutually exclusive, the probability of either/or event is the sum of the probabilities of each event. However, when two events are independent, the probability of both events occurring is the product of the probabilities of each event:

	P(A∪B)	P(A∩B)
Mutually exclusive	P(A) + P(B)	$\varnothing$
Independent	?	P(A) P(B)

Mutually Exclusive & Independent?

- Recall that if events A and B are mutually exclusive, then P(A∩B) = 0. So if P(B) ≠ 0, then by the definition of conditional probability we have: $P\left(A\middle|B\right) = \dfrac{P\left(A \cap B\right)}{P\left(B\right)} = 0$

- Now we ask, can events A and B be both independent and mutually exclusive?

- Suppose, they were then P(A|B) = P(A) and P(A|B) = 0, and thus P(A) = 0. Furthermore, P(B|A) = P(B) and P(B|A) = 0, and thus P(B) = 0.

- Therefore, if events A and B are both independent and mutually exclusive, then P(A) = 0 and P(B) = 0. That is, events that are both mutually exclusive and independent are only events with zero probability.

Example 12.4

Two dice are tossed, one at a time. Let A = "6 on first die", B = "sum of 7", and C = "sum of 8". Which events are independent? Which events are mutually exclusive?

Solution:

B&C: Clearly, B and C are mutually exclusive with non-zero probabilities, and therefore not independent.

To answer the question for A&B and A&C, we need the following probabilities:

There are six possible outcomes on a die and only one way to roll a 6. Thus, **P(A) = 1/6** . In rolling two dice, there are 6 × 6 = 36 possible outcomes. There are six ways to get a "sum of 7", {(16), (25), (34), (43), (52), (61)}. Thus, **P(B) = 6/36 = 1/6**. There are five ways to get a "sum of 8", {(26), (35), (44), (53), (62)}. Thus, **P(C) = 5/36**. Event (A∩B) can only occur with a 6 on the first die and a 1 on the second die. Thus, **P(A∩B) = 1/36**. Event (A∩C) can only occur with a 6 on the first die and a 2 on the second die. Thus, **P(A∩C) = 1/36**.

Example 12.4

And we have the following:

A&B: $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/36}{6/36} = \dfrac{1}{6} = P(A)$

Thus, A&B are independent (and not mutually exclusive.)

A&C: $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{1/36}{5/36} = \dfrac{1}{5} \neq P(A)$

Thus, A&C are not independent (and not mutually exclusive.)

When to Assume Independence?

When is it reasonable to assume independence? Some examples relevant to our chapter:

1. Result of second sample is independent of first if first is placed back into sampling pool.

2. Sex of second child is independent of first child.

3. Random selection of alleles on genes located on separate chromosomes (Mendel's Second Law of Independent Assortment).

4. The genotypes of non-related individuals are independent events.

Example 12.5 (Beetle Sampling with Replacement)

Suppose we have a population of 150 beetles. Forty-five of the beetles have wings and the rest are wingless. You select a beetle at random and record whether or not it has wings, and then put it back amongst the other beetles. Then you select another beetle. Let events A = "first beetle has wings" and B = "second beetle has wings"

What is the probability that (a) the first and second beetle have wings? and (b) the first beetle has wings and the second beetle does not have wings?

Solution:

(a) $P(A \cap B) = P(A)P(B) = 0.3 \times 0.3 = 0.09$

(b) $P(A \cap \overline{B}) = P(A)P(\overline{B}) = 0.3 \times 0.7 = 0.21$

Example 12.6 (Blood Typing)

Suppose Jacob has O+ blood with genotype OO/Rr, and Anna has B– blood with genotype BO/rr. What is the probability that a child of Jacob and Anna will have (a) B+ blood, and (b) O– blood?

Solution: We start by making a Punnett square for each gene:

♂	♀ B	♀ O
O	OB	OO
O	OB	OO

♂	♀ r	♀ r
R	Rr	Rr
r	rr	rr

Thus:

(a) $P(B+) = P(B)P(+) = 0.5 \times 0.5 = 0.25$

(b) $P(O-) = P(O)P(-) = 0.5 \times 0.5 = 0.25$

Example 12.7 (Tay Sachs Disease)

This example uses both conditional probability and independence.

Jack and Judy each had brothers afflicted with Tay-Sachs disease. From Example 12.1, the probability of being a carrier if your sibling has Tay-Sachs disease is 2/3. What is the probability that Jack and Judy have a child with Tay-Sachs disease?

Solution: The only way the child can be tt is if Jack and Judy are both Tt. Let events A = "Judy is Tt", B = "Jack is Tt", C = "child is tt". Thus:

$$P\big(C \cap (A \cap B)\big) = P\big(C \big| (A \cap B)\big) P(A \cap B) = P\big(C \big| (A \cap B)\big) P(A) P(B)$$

$$= \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{9}$$

Example 12.8 (Family Planning)

Assume the births of boys and girls are equiprobable and independent. Let events A = "first child is a girl," B = "second child is a girl," and C = "third child is a girl." These are all independent events.

(a) What is the probability that in a family with two children, the children are both girls?

$$P(A \cap B) = P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

(b) What is the probability that in a family with three children, the children are all girls?

$$P(A \cap B \cap C) = P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Example 12.8 (Family Planning)

(c)   What is the probability that in a family with six children, the children are all the same sex?

The probability that all six children are the same sex equals the probability they are all boys plus the probability they are all girls. Let events A_1, A_2, . . . , A_6 be the events that the first, second, . . . , sixth (respectively) child is a girl. Then:

$$P\left(A_1 \cap A_2 \cap \cdots \cap A_6\right) = P\left(A_1\right)P\left(A_2\right)\cdots P\left(A_6\right) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$P\left(\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_6\right) = P\left(\overline{A}_1\right)P\left(\overline{A}_2\right)\cdots P\left(\overline{A}_6\right) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$P\left(\text{all same sex}\right) = \frac{1}{64} + \frac{1}{64} = \frac{1}{32}$$

Example 12.8 (Family Planning)

(d) What is the probability that in a family with six children, there is at least one girl?

Let A = "at least one girl". Then Ā = "no girls", i.e. all boys. Then:

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{64} = \frac{63}{64} \approx 0.9844$$

Homework

Chapter 12: 1ab, 2, 4, 5, 6, 7, 9

Some answers: