

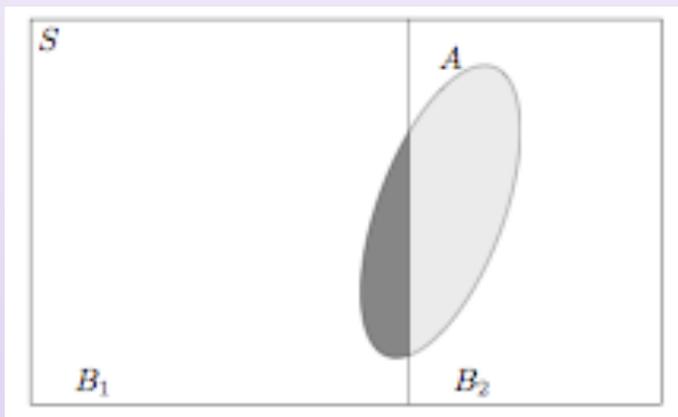
Chapter 13: Sequential Experiments & Bayes' Theorem

1. (13.1) Partition Theorem
2. (13.2) Bayes' Theorem

1. (13.1) Partition Theorem

Probability Law: Partition Theorem

Suppose we have a sample space S that is divided into two regions B_1 and B_2 , such that these two regions are mutually exclusive and $B_1 \cup B_2 = S$:



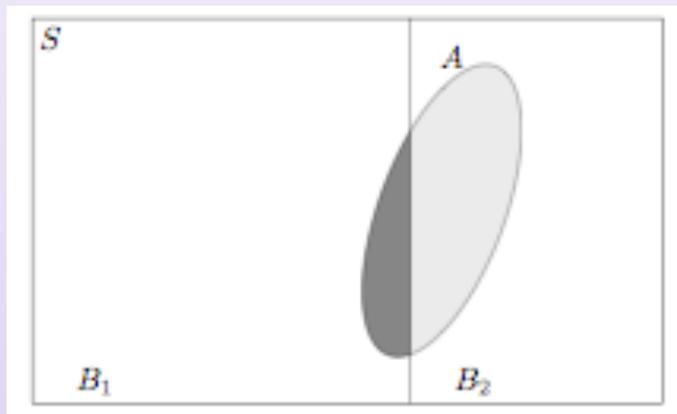
This is known as a **partition** of the sample space. To find the probability of some event A in this sample space S , we use the Partition Theorem.

1. (13.1) Partition Theorem

Probability Law: Partition Theorem

If B_1 and B_2 form a partition of the sample space S , and A is an event in S then:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$



We can think of the partition theorem as, “The probability of event A is the probability of event A given event B_1 weighted by the probability of event B_1 plus the probability of event A given event B_2 weighted by the probability of event B_2 ”

1. (13.1) Partition Theorem

Example 13.1 (Albinism)

Both of Jay's parents are carriers for albinism. Jay's sister has the disease, but Jay does not (though it is unknown if he is a carrier or not). Jay's wife, Mary, has no history of albinism in her family so we can assume she is homozygous dominant with respect to the albinism gene. What is the probability that Jay and Mary's child will be a carrier of the disease?

Solution: It is helpful to use a human pedigree chart. The notation for such a chart is shown here:

Symbol	Meaning
□	Male homozygous dominant
○	Female homozygous dominant
■	Male homozygous recessive
●	Female homozygous recessive
◻	Male heterozygous
◐	Female heterozygous

Table 14.1: Human Pedigree Notation

1. (13.1) Partition Theorem

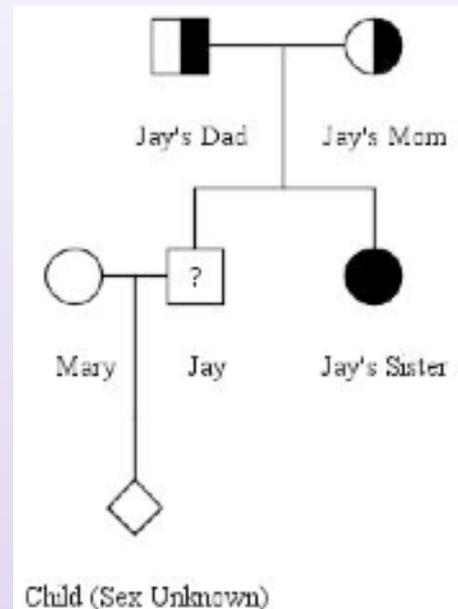
Example 13.1 (Albinism)

- Thus, the chart for our problem is:
- Let event A = “child is a carrier.”
- To find $P(A)$ we can partition the sample space into:
 - B_1 = “Jay is heterozygous” and
 - B_2 = “Jay is homozygous dominant.”
- Given that Jay’s mother and father are both heterozygous, the Punnett square showing the possible genotypes for Jay is:

		♀	
		A	a
♂	A	AA	Aa
	a	aA	aa

Since Jay is not albino, \Rightarrow

$$P(B_1) = \frac{2}{3}$$
$$P(B_2) = \frac{1}{3}$$



1. (13.1) Partition Theorem

Example 13.1 (Albinism)

Given that Jay is heterozygous, the Punnett square displaying the possible genotypes for a child of Jay and Mary's is:

		♀	
		A	A
♂	A	AA	Aa
	a	aA	aa

Thus: $P(A|B_1) = \frac{1}{2}$

Given that Jay is homozygous dominant, the Punnett square displaying the possible genotypes for their child is:

		♀	
		A	A
♂	A	AA	AA
	A	AA	AA

Thus: $P(A|B_2) = 0$

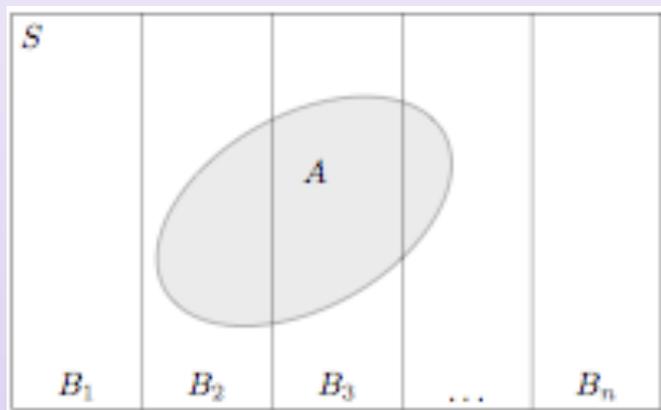
And by the Partition Theorem:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) = \frac{1}{2} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3}$$

1. (13.1) Partition Theorem

Probability Law: Generalized Partition Theorem

To generalize the partition theorem, suppose the sample space S is partitioned into n regions, B_1, B_2, \dots, B_n such that each B_i is mutually exclusive of the others, and $B_1 \cup B_2 \cup \dots \cup B_n = S$, and let A be an event in S :



$$\text{Then: } P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_n)P(B_n)$$

1. (13.1) Partition Theorem

Example 13.2 (Pedigree)

Suppose we know the genotypes of Billy's maternal grandparents for a certain genetic disease are Aa and Aa , and the genotype of Billy's father is AA (no disease). What is the probability that Billy is a carrier for the disease?

Solution: Using a Punnett square, we can show that there are three possibilities for the genotype of Billy's mother, and we can determine the probabilities for these possibilities:

		♀	
		A	a
♂	A	AA	Aa
	a	aA	aa

$$B_1 = \text{"mom is } Aa\text{"} \quad P(B_1) = 1/2$$

$$\Rightarrow B_2 = \text{"mom is } aa\text{"} \Rightarrow P(B_2) = 1/4$$

$$B_3 = \text{"mom is } AA\text{"} \quad P(B_3) = 1/4$$

It is essential to note that these are three mutually exclusive events that partition the sample space.

1. (13.1) Partition Theorem

Example 13.2 (Pedigree)

Now let event A = “Billy a carrier (Aa)”. We can use the generalized partition theorem to find $P(A)$:

		♀	
		A	A
♂	A	AA	AA
	a	aA	aA

		♀	
		A	A
♂	a	aA	aA
	a	aA	aA

		♀	
		A	A
♂	A	AA	AA
	A	AA	AA

Known:

$$P(B_1) = 1/2$$

$$P(B_2) = 1/4$$

$$P(B_3) = 1/4$$

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

$$= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{1}{2}$$

Thus, the probability that Billy is a carrier of the disease is 50%.

1. (13.1) Partition Theorem

Example 13.2 Tay-Sachs?

How would Example 13.2 been different if we had considered a disease like Tay-Sachs disease? Recall, that individuals having Tay-Sachs disease usually do not live beyond the age of four. Thus, the Punnett square showing the possible genotypes for Billy's mother would be:

		♀	
		T	t
♂	T	TT	Tt
	t	tT	tt

$$B_1 = \text{"mom is Tt"} \quad P(B_1) = 2/3$$

$$\Rightarrow B_2 = \text{"mom is tt"} \Rightarrow P(B_2) = 0$$

$$B_3 = \text{"mom is TT"} \quad P(B_3) = 1/3$$

Now let event $A = \text{"Billy a carrier (Tt)"}.$ We have:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

$$= \frac{1}{2} \cdot \frac{2}{3} + 1 \cdot 0 + 0 \cdot \frac{1}{3} = \frac{1}{3}$$

2. (13.2) Bayes' Theorem

Probability Law: Bayes' Theorem

Suppose we know the conditional probability $P(A|B)$. We would like a way to find $P(B|A)$. The mathematical tool we have for this is known as Bayes' Theorem.

If B_1 and B_2 form a partition of the sample space S , and A is an event in S then:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)}, \text{ where } P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$

This follows directly from three of the previous probability laws:

$$\begin{aligned} P(B_1|A) &= \frac{P(B_1 \cap A)}{P(A)} = \frac{P(A \cap B_1)}{P(A)} \\ &= \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} \end{aligned}$$

2. (13.2) Bayes' Theorem

Example 13.3 (Head Injuries)

Among individuals who have sustained head injuries, x-rays reveal that only about 6% have skull fractures. Nausea is a standard symptom of a skull fracture and occurs in 98% of all skull fracture cases. With other types of head injuries, nausea is present in about 70% of all cases. An individual has just suffered a head injury is not nauseous. Find the probability that he has a skull fracture.

Solution: Let events F = "skull fracture" and N = "nausea". Since only 6% of head injuries result in skull fractures $P(F) = 0.06$ and $P(\bar{F}) = 0.94$. Given individuals who have skull fractures, 98% have nausea. Give those who do not have skull fractures, 70% have nausea. Thus, $P(N|F) = 0.98$ and $P(N|\bar{F}) = 0.70$. We seek $P(F|\bar{N})$.

2. (13.2) Bayes' Theorem

Example 13.3 (Head Injuries)

Applying Bayes' Theorem:

Given:

$$P(F) = 0.06 \quad P(\bar{F}) = 0.94$$

$$P(N|F) = 0.98 \quad P(N|\bar{F}) = 0.70$$

$$\begin{aligned} P(F|\bar{N}) &= \frac{P(\bar{N}|F)P(F)}{P(\bar{N}|F)P(F) + P(\bar{N}|\bar{F})P(\bar{F})} \\ &= \frac{(1 - 0.98)(0.06)}{(1 - 0.98)(0.06) + (1 - 0.7)(0.94)} = 0.004 \end{aligned}$$

Thus the probability that a person having just suffered a head injury and having no nausea has a skull fracture is very unlikely (less than 1%).

2. (13.2) Bayes' Theorem

Example 13.4 (Drug Testing)

Suppose that a drug test for an illegal drug is such that it is 98% accurate in the case of a user of that drug (i.e. it produces a positive result with probability 0.98 in the case that the tested individual uses the drug) and 90% accurate in the case of a non-user of the drug (i.e. it is negative with probability 0.90 in the case the person does not use the drug). Suppose it is known that 10% of the entire population uses this drug.

Let events $+$ = “drug test is positive”, $-$ = “drug test is negative”, and A = “the person tested has the drug in his or her system”.

$$\text{Then: } P(A) = 0.10 \quad P(\bar{A}) = 0.90$$

$$P(+|A) = 0.98 \quad P(-|\bar{A}) = 0.90$$

2. (13.2) Bayes' Theorem

Example 13.4 (Drug Testing)

- (a) What is the probability of a false positive with this test (i.e. the probability of obtaining a positive drug test given that the person tested is a non-user)?

Solution:
$$P(+|\bar{A}) = 1 - P(-|\bar{A})$$
$$= 1 - 0.90 = 0.10$$

Given:

$$P(A) = 0.10 \quad P(\bar{A}) = 0.90$$

$$P(+|A) = 0.98 \quad P(-|\bar{A}) = 0.90$$

- (b) What is the probability of obtaining a false negative for this test?

Solution:
$$P(-|A) = 1 - P(+|A)$$
$$= 1 - 0.98 = 0.02$$

2. (13.2) Bayes' Theorem

Example 13.4 (Drug Testing)

(c) You test someone and the test is positive. What is the probability that the tested individual uses this illegal drug?

Solution:

$$P(A|+) = \frac{P(+|A)P(A)}{P(+)}$$

$$= \frac{P(+|A)P(A)}{P(+|A)P(A) + P(+|\bar{A})P(\bar{A})}$$

$$= \frac{(0.98)(0.10)}{(0.98)(0.10) + (0.10)(0.90)} = 0.521$$

Given/Known:

$$P(A) = 0.10 \quad P(\bar{A}) = 0.90$$

$$P(+|A) = 0.98 \quad P(-|\bar{A}) = 0.90$$

$$P(+|\bar{A}) = 0.10 \quad P(-|A) = 0.02$$

Thus, the probability that the person who tested positive actually has the drug within their system is 52.1%

2. (13.2) Bayes' Theorem

Sensitivity & Specificity

- The **sensitivity** of a medical test is defined as the probability that a test will be positive given that a person has the disease for which they are being tested.
- The **specificity** of a medical test is the probability that a test will be negative given that the person does not have the disease.
- Thus, in the previous example, the sensitivity is $P(+ | A) = 0.98$ and the specificity is $P(- | \bar{A}) = 0.90$.
- Let us look at another medical testing scenario where there is the possibility of false positives and false negatives.

2. (13.2) Bayes' Theorem

Example 13.5 (Genetic Testing)

Suppose there is a disease caused by the presence of two recessive alleles aa , and that a genetic test has been developed to determine whether a normal individual is a carrier Aa or a non-carrier AA .

Suppose among known carriers, the test correctly identifies the presence of the a allele 80% of the time. $P(+ | Aa) = 0.80$

Among known AA individuals, the test indicates the presence of the a allele 20% of the time. $P(+ | AA) = 0.20$

What is the probability that a person is a carrier if both of her parents are carriers given they get a positive result (i.e. yes presence of allele a)? $P(Aa | +) = ???$

2. (13.2) Bayes' Theorem

Example 13.5 (Genetic Testing)

Solution: We can assume if a person is taking this test then they do not have the disease, i.e. the person does not have genotype aa .

Let event AA = “person is homozygous dominant”, Aa = “person is heterozygous”, $+$ = “test result is positive (carrier)”, and $-$ = “test result is negative (non-carrier)”. We seek $P(Aa | +)$.

Since the tested individual's parents are both carriers the Punnett square showing the possible genotypes for the individual is:

		♀	
		A	a
♂	A	AA	Aa
	a	aA	aa

$$\Rightarrow \begin{aligned} P(Aa) &= \frac{2}{3} \\ P(AA) &= \frac{1}{3} \end{aligned}$$

2. (13.2) Bayes' Theorem

Example 13.5 (Genetic Testing)

Notice we can partition the sample space into the mutually exclusive events AA and Aa. Applying Bayes' Theorem, we find that:

Given/Known:

$$P(+|Aa) = 0.80 \quad P(+|AA) = 0.20$$

$$P(Aa) = \frac{2}{3} \quad P(AA) = \frac{1}{3}$$

$$\begin{aligned} P(Aa|+) &= \frac{P(+|Aa)P(Aa)}{P(+)} = \frac{P(+|Aa)P(Aa)}{P(+|Aa)P(Aa) + P(+|AA)P(AA)} \\ &= \frac{(0.80)(2/3)}{(0.80)(2/3) + (0.20)(1/3)} = 0.889 \end{aligned}$$

Thus, the probability that the individual who tested positive is a carrier is 88.9%.

Homework

Chapter 13: 1-14