

Lectures 4a,4b: Exponential & Logarithmic Functions

1. (4.1) Exponential & Logarithmic Functions in Biology
2. (4.2) Exponential & Logarithmic Functions: Review
3. (4.3) Allometry
4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

1. (4.1) Exponential & Logarithmic Functions in Biology

- Recall from last time that we were able to come up with a “best” linear fit for bivariate data that seem to be linearly related
- In many cases, however, if you were to plot data points obtained from biological measurements you would find that the data are not linearly related
- As an example, consider population growth:
 - Imagine algae growing in a Petri dish, starting from a single cell
 - Through time the cell will split (or else die), then each new cell will split again, etc.
 - The total number of cells does not increase linearly (in an additive manner) through time *but* multiplicatively (by doubling)
 - If you were to plot the number of cells through time, it would **not** increase linearly

1. (4.1) Exponential & Logarithmic Functions in Biology

The Key Idea

- **Exponentials** are used to describe something that increases (or decreases) in a multiplicative manner
- **Logarithms** are a way to rescale something that is increasing (or decreasing) in a multiplicative manner *so as to measure its increase in a new way that **does** increase (or decrease) linearly*
- This arises from the fact that *the logarithm of a product is the sum of the logarithms* of the components of the product:

$$\log(ab) = \log(a) + \log(b)$$

$$\ln(ab) = \ln(a) + \ln(b)$$

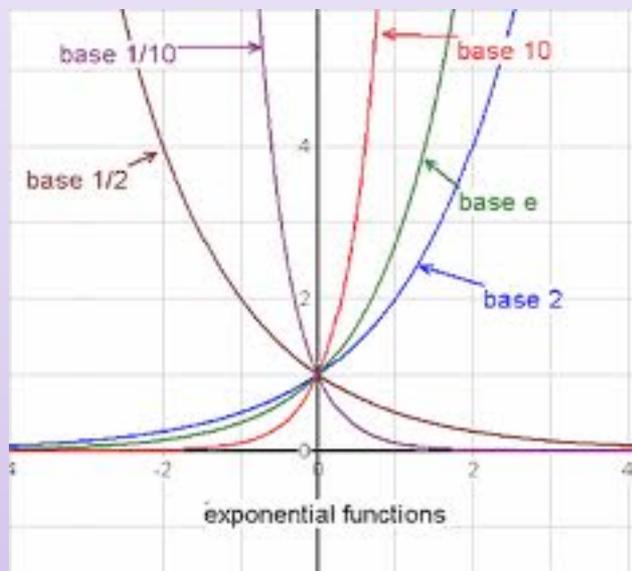
2. (4.2) Exponential & Logarithmic Functions: Review

Exponential Functions

- **Definition:** An **exponential function** is a function of the form $f(x) = a^x$ where $a > 0$ is a fixed real number called the base of the exponent
 - Note: This is different from the power function $f(x) = x^a$ where the base is the variable and the exponent a is a fixed constant

$$0 < a < 1$$

exponential decay



$$a > 1$$

exponential growth

2. (4.2) Exponential & Logarithmic Functions: Review

Laws of Exponents

1. $a^x a^y = a^{x+y}$

2. $\frac{a^x}{a^y} = a^{x-y}$

3. $(a^x)^y = a^{xy}$

4. $(ab)^x = a^x b^x$

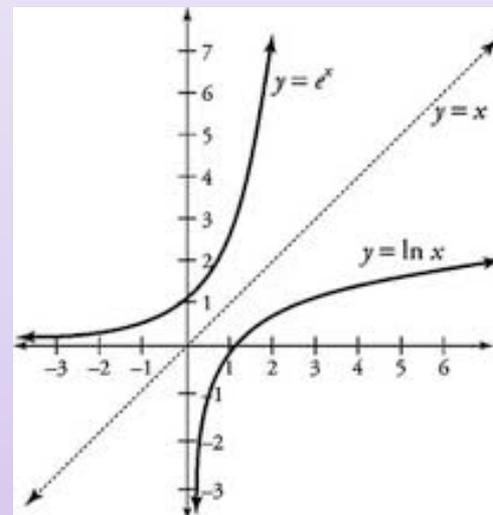
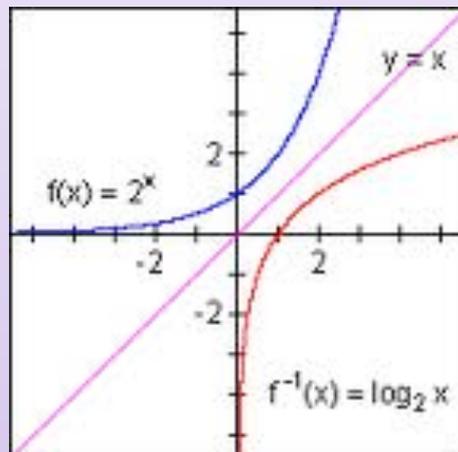
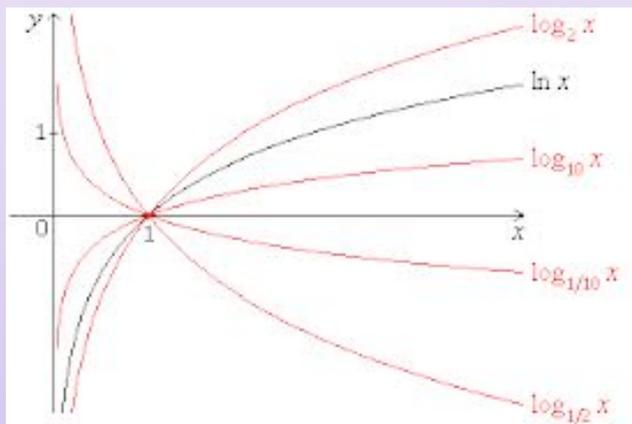
5. $a^0 = 1$

6. $\frac{1}{a^x} = a^{-x}$

2. (4.2) Exponential & Logarithmic Functions: Review

Logarithmic Functions

- **Definition:** A **logarithmic function** is a function of the form $f(x) = \log_a x$ where $a > 0$ is a fixed real number called the base of the logarithm
 - The logarithmic function is the inverse of the exponential function:
 $\log_a(a^x) = x$ and $a^{\log_a x} = x$



2. (4.2) Exponential & Logarithmic Functions: Review

Laws of Logarithms

$$1. \quad \log_a(xy) = \log_a x + \log_a y$$

$$2. \quad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$3. \quad \log_a x^k = k \log_a x$$

$$4. \quad \log_a a = 1 \quad \text{and} \quad \log_a 1 = 0$$

$$5. \quad \log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a} = \frac{\log_b x}{\log_b a}$$

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.1

Solve: $5^x = 0.23$

Solution: $\ln 5^x = \ln 0.23$

$$x \ln 5 = \ln 0.23$$

$$x = \frac{\ln 0.23}{\ln 5}$$

$$x \approx -0.91$$

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.2

If $\log_a x = 2.1$ and $\log_a y = 0.45$, then find $\log_a(x^3 y)$.

Solution:

$$\begin{aligned}\log_a(x^3 y) &= \log_a x^3 + \log_a y \\ &= 3\log_a x + \log_a y \\ &= 3(2.1) + 0.45 \\ &= 6.75\end{aligned}$$

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.3

Solve: $\log_4(3x) = 1.4$

Solution: $4^{\log_4(3x)} = 4^{1.4}$

$$3x = 4^{1.4}$$

$$x = \frac{4^{1.4}}{3}$$

$$x \approx 2.3$$

2. (4.2) Exponential & Logarithmic Functions: Review

Half-Life & Doubling Time

- Logarithms and exponentials are frequently used to calculate the half-life of radioactive substances or the doubling time of populations.
- The **half-life** of a radioactive substance is how long it takes for N amount of that substance to decay to $1/2 N$ amount
- The **doubling time** of a population is how long it takes N individuals in that population to reproduce and become $2N$ individuals

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.4

- It has been shown that the amount of drug in a person's body decays exponentially
- Let $C(t)$ denote the concentration of the drug in the bloodstream at time t (in days), and let C_0 be the initial amount of drug in the bloodstream
- Then we can write $C(t) = C_0 e^{-kt}$ where $k > 0$ is known as a decay constant. The larger the value of k , the more quickly the drug decays in the bloodstream
- (a) If the drug has a half-life of 10 days, what is the value of k ?
- (b) What percent of the administered amount of drug remains in the bloodstream after 4 hours?

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.4

- (a) A half-life of 10 days means that when $t = 10$ we should have $1/2$ of the initial amount of drug, C_0 :

$$C(10) = \frac{1}{2}C_0$$

- From the given relation we have:

$$C(10) = C_0e^{-10k}$$

- Hence:

$$0.5C_0 = C_0e^{-10k}$$

$$0.5 = e^{-10k}$$

$$\ln 0.5 = \ln e^{-10k}$$

$$\ln 0.5 = -10k$$

$$0.069 \approx \frac{-1}{10} \ln 0.5 = k$$

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.4

- (b) Using the results from (a), our equation for the decay of the drug in the bloodstream is $C(t) = C_0 e^{-0.069t}$
- Recall that t is measured in days, thus if we want to know how much is left after 4 hours, we need to convert this to units of days: $4 \text{ hours} \times 1 \text{ day}/24 \text{ hours} = 1/6 \text{ days}$. Thus, we want to know how much of C_0 is left after $1/6$ days:

$$\begin{aligned} C(1/6) &= C_0 e^{-0.069(1/6)} \\ &\approx 0.989C_0 \end{aligned}$$

- Thus, there is approximately 98.9% of the administered dose left after 4 hours.

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.5

- Biologists studying salmon have found that the oxygen consumption of yearling salmon (in appropriate units) increases exponentially with the speed of swimming
- Specifically, $f(x) = 100e^{0.6x}$ where x is the speed in feet per second, and f gives the oxygen consumption
- Find each of the following:
 - (a) The oxygen consumption when the fish are not moving
 - (b) The oxygen consumption at a speed of 2 ft/s
 - (c) If a salmon is swimming at 2 ft/s, how much faster does it need to swim to double its oxygen consumption?

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.5

(a) Find the oxygen consumption when the fish are not moving.

Solution:

- When the fish are not moving, their speed is 0 ft/s.
- Thus, when $x=0$, $f(0) = 100e^{0.6 \times 0}$
$$= 100e^0$$
$$= 100 \times 1$$
$$= 100$$
- Thus, when the fish are not moving, their oxygen consumption is 100 units.

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.5

(b) Find the oxygen consumption at a speed of 2 ft/s.

Solution:

- Here $x=2$ and $f(2) = 100e^{0.6 \times 2}$
 $= 100e^{1.2}$
 ≈ 332
- Thus, at a speed of 2 ft/s, oxygen consumption is 332 units.

2. (4.2) Exponential & Logarithmic Functions: Review

Example 4.5

(c) If a salmon is swimming at 2 ft/s, how much faster does it need to swim to double its oxygen consumption?

Solution:

- We want to solve for x when oxygen consumption is $2 \times f(2) = 2 \times 332 = 664$ units.

$$664 = 100e^{0.6x}$$

$$6.64 = e^{0.6x}$$

$$\ln 6.64 = \ln e^{0.6x}$$

$$\ln 6.64 = 0.6x$$

$$x = \ln 6.64 / 0.6 \approx 3.16 \text{ ft/s}$$

- Notice, 3.16 ft/s is $3.16 - 2 = 1.16$ ft/s faster than 2 ft/s. Thus, to get double the oxygen consumption as when swimming 2 ft/s, the salmon would have to swim 1.16 ft/s faster.

3. (4.3) Allometry

Introduction

- Definition: two variables x and y are said to be **allometrically related** if $y = ax^b$ where a and b are real constants
- Note: This is different from an exponential relationship where the variable is in the exponent: $y = a^x$ It is important that you know the difference, especially in what follows
- Some typical biological relationships that are allometric:
 - Length vs. Volume
 - Surface area vs. Volume
 - Body weight vs. brain weight
 - Body weight vs. blood volume

3. (4.3) Allometry

Example 4.8 (Elephants)

- It has been determined that for any elephant, surface area of the body can be estimated as an allometric function of trunk length:

$$\text{surface area} = (\text{some number}) \times (\text{trunk length})^{(\text{some number})}$$

- For African elephants the allometric exponent is 0.74:

$$y = ax^{0.74}$$

- If a particular elephant has a surface area of 200 ft² and a trunk length of 6 ft, what is the expected surface area of an elephant with a trunk length of 7 ft?

3. (4.3) Allometry

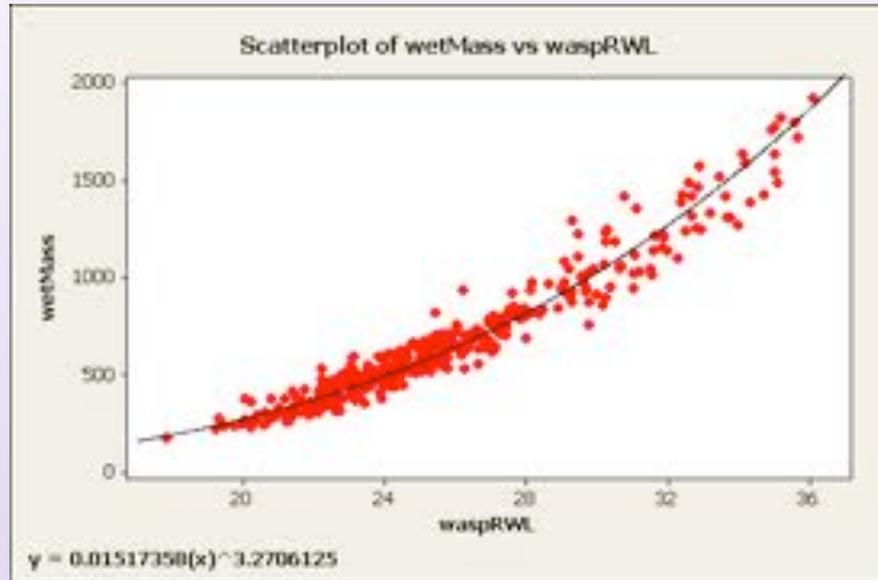
Example 4.8 (Elephants)

- We are given:
 - SA is allometrically related to TL: $y = ax^b$
 - The allometric exponent: $y = ax^{0.74}$
 - A data point: $(x, y) = (6, 200)$
- We need to use the given data point to solve for yet unknown constant, a , and then use the model to make the requested prediction.
 $(x, y) = (6, 200) \Rightarrow 200 = a \cdot 6^{0.74} \Rightarrow a \approx 53.05$
- Hence the equation: $y = 53.05x^{0.74}$
- And the prediction $y = 53.05 \cdot 7^{0.74} \approx 223.9$
- expected body surface area for a trunk 7 ft long is 223.9 ft²

4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Overview

Mass Vs. Right Wing Length
Non-Linear Model for Wasps



- As pointed out above, data often appear to have a strong relationship, but that relationship is not linear
- We would like to apply the same analysis as before, but we would need to develop a new method to get the best fit curve
- In some situations, however, a *rescaling* of the data could transform it in such a way that the new relationship is linear

4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Rewriting Equations

- To see why this is the case, we begin- not with data- but with the types of curves we use to model the data
- We start with an exponential equation:

$$y = c \cdot a^x$$

$$\ln(y) = \ln(c \cdot a^x)$$

$$\ln(y) = \ln c + \ln a^x$$

$$\ln(y) = \ln c + x \ln a$$

$$\ln(y) = (\ln a)x + \ln c$$

- Let's compare the structure of this equation with the equation of a line

4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Rewriting Equations

$$\ln(y) = (\ln a)x + \ln c$$

$$\begin{array}{l} \text{dependent} \\ \text{variable} \end{array} = \begin{array}{l} \text{constant} \times \\ \text{independent} \\ \text{variable} \end{array} + \text{constant}$$

$$y = mx + b$$

The new equation is a linear equation.

Rewriting Equations

- Now consider an allometric equation:

$$y = a \cdot x^b$$

$$\ln(y) = \ln(ax^b)$$

$$\ln(y) = \ln a + \ln x^b$$

$$\ln(y) = \ln a + b \ln x$$

$$\ln(y) = b \ln x + \ln a$$

- Again, we obtain a linear equation.
- We need to take a closer look at these transformations.

Rescaling Data

Exponential

$$y = c \cdot a^x$$

$$\ln(y) = (\ln a)x + \ln c$$

Allometric

$$y = a \cdot x^b$$

$$\ln(y) = b \ln x + \ln a$$

- What we see is that if we have data that are exponentially related, and we rescale the y coordinates of the data by taking their logarithm $(x,y) \rightarrow (x, \ln y)$, then the scatter plot of the rescaled data will be linear
- If we have data that are allometrically related, and we rescale the x and y coordinates of the data by taking their logarithm $(x,y) \rightarrow (\ln x, \ln y)$, then the scatter plot of the rescaled data will be linear

4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

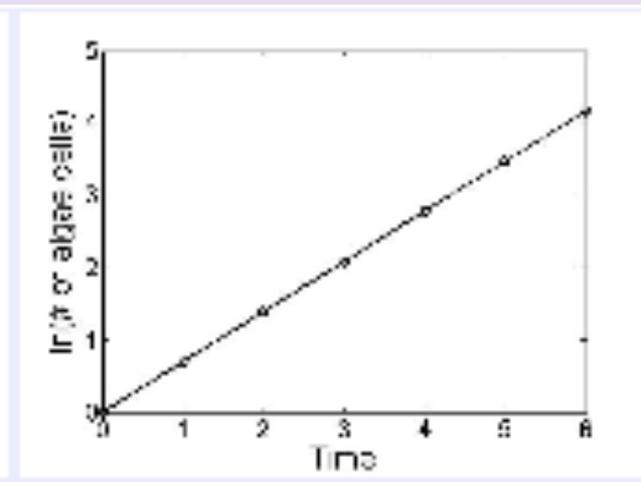
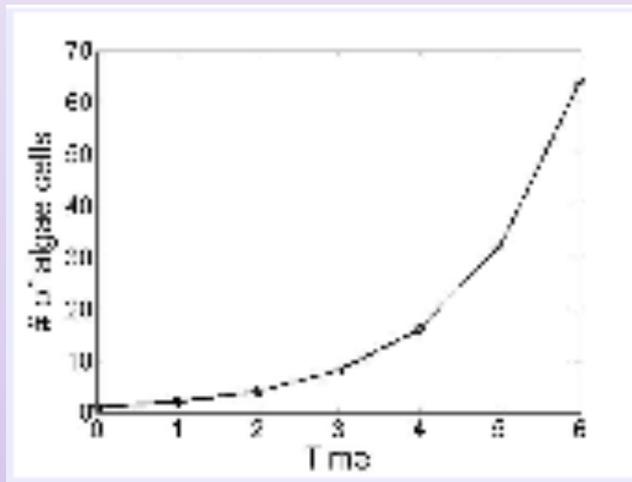
Example 4.9 (Algae Growth)

- Imagine algae growing in a Petri dish, starting from a single cell. After some time the cell will split (or else die). Then, each of the two cells will split, and so on. Thus, the number of cells in the Petri dish does not increase linearly (i.e. in an additive manner), but multiplicatively (by doubling each time).
- Suppose the doubling time is 1 day. Thus if we start off with one cell, after 1 day we will have 2 cells, after 2 days 4 cells, after 3 days 8 cells, etc.
- Let us see what happens when we rescale the y-axis data:

4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Example 4.9 (Algae Growth)

x -axis	y -axis	y -axis rescaled
Time t	# of cells N	$\ln N$
0	1	0
1	2	0.693
2	4	1.386
3	8	2.079
4	16	2.773
5	32	3.466
6	64	4.159



4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Example 4.11 (Mutation Rates)

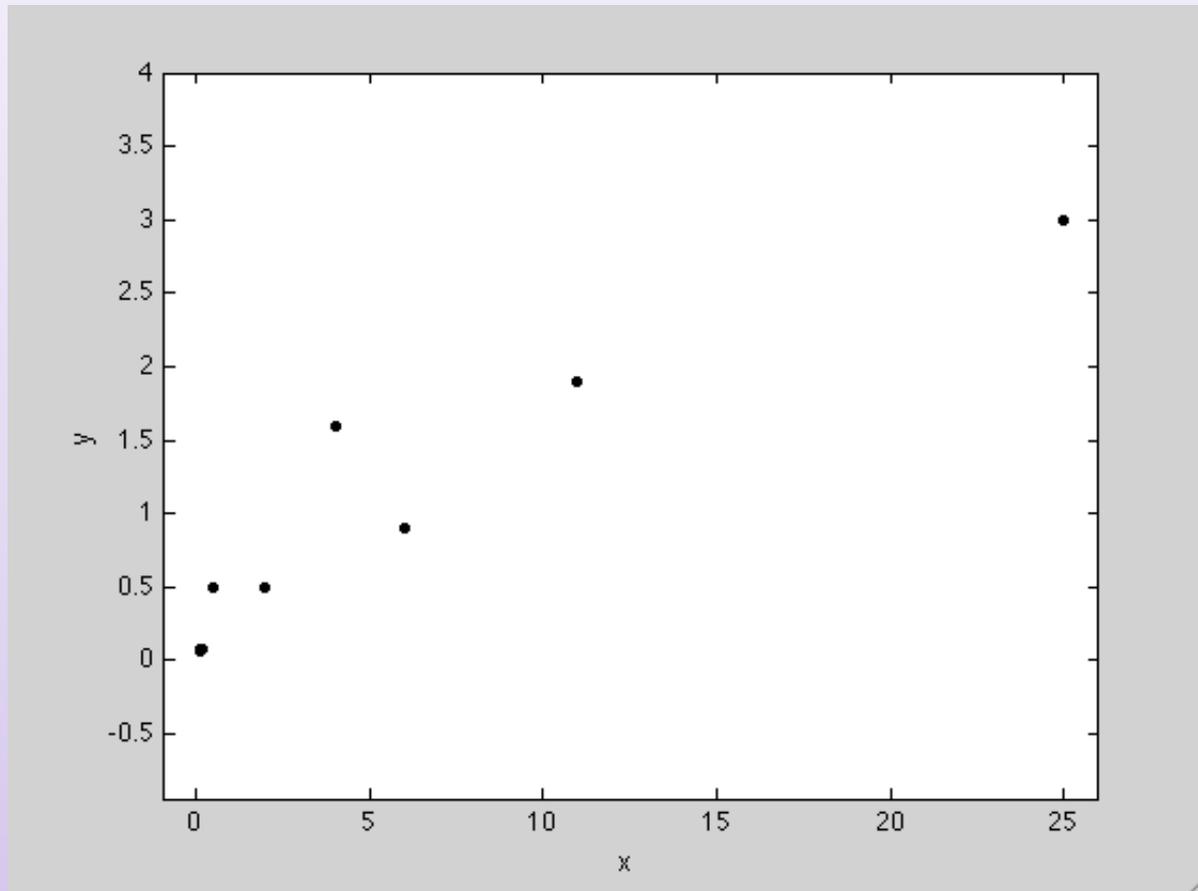
- Researchers studying the relationship between the generation time of a species and the mutation rate for genes that cause deleterious effects gathered the following data:

Species	Generation Time (in years)	Genomic Mutation Rate (per generation)
<i>D. melanogaster/D. pseudoobscura</i>	0.1	0.070
<i>D. melanogaster/D. simulans</i>	0.1	0.058
<i>D. picticornis/D. silvestris</i>	0.2	0.071
Mouse/rat	0.5	0.50
Chicken/old world quail	2	0.49
Dog/cat	4	1.6
Sheep/cow	6	0.90
Macaque/New World Monkey	11	1.9
Human/chimpanzee	25	3.0

4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Example 4.11 (Mutation Rates)

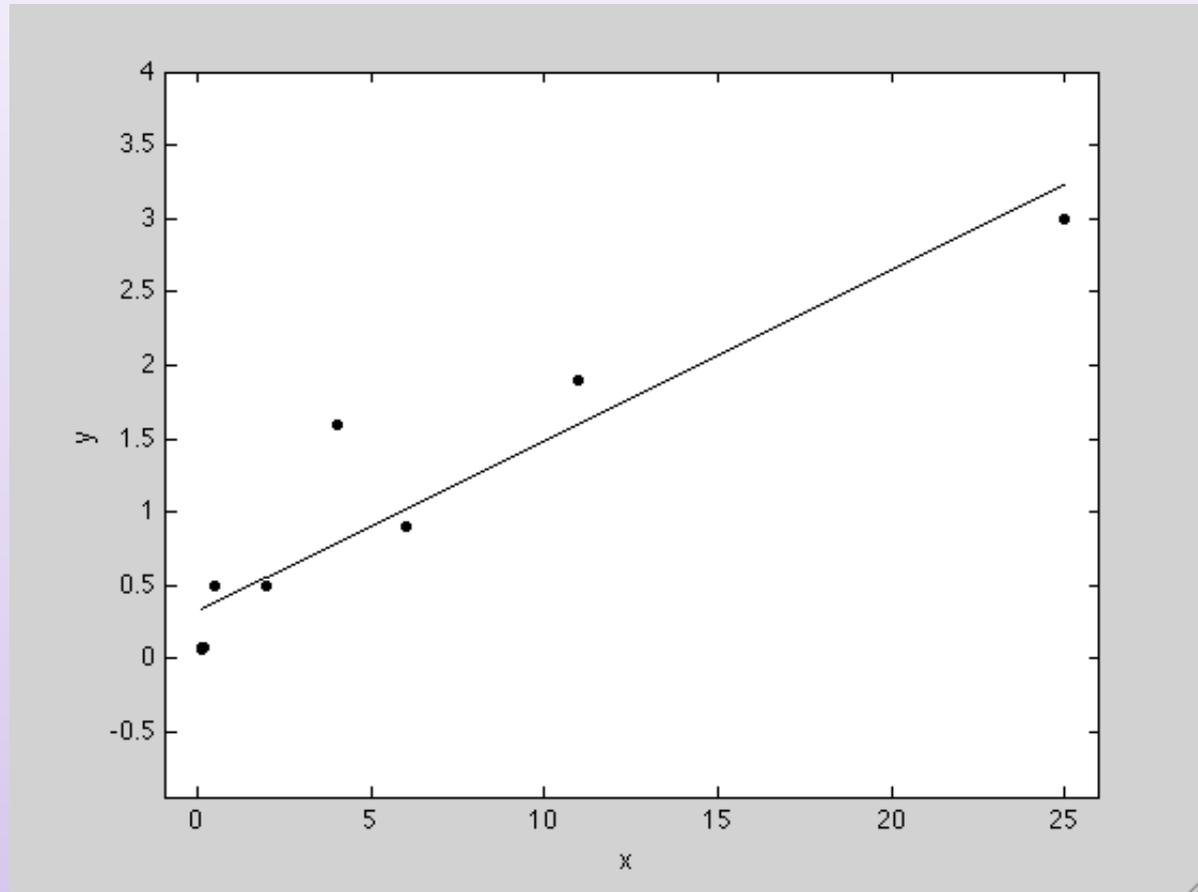
- A scatterplot of the data:



4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Example 4.11 (Mutation Rates)

- A scatterplot of the data with LSR:



4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Example 4.11 (Mutation Rates)

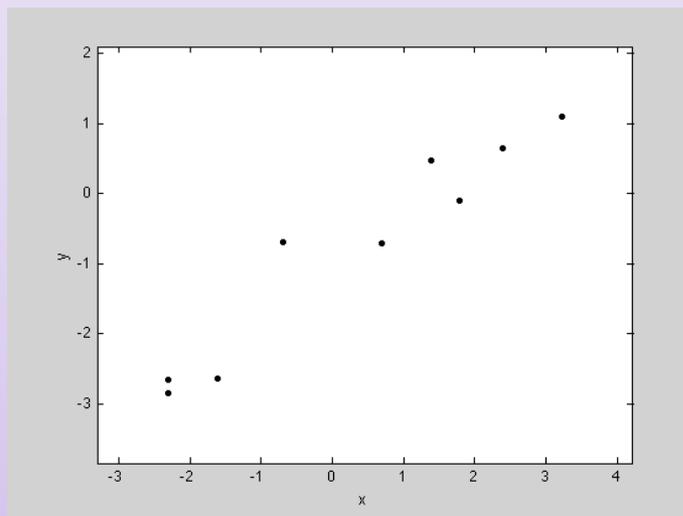
- Is this a good fit?
- The MATLAB output:
 - Eqn for LSR: $y = 0.116079 x + 0.323640$
 - $\rho = 0.934107$
 - The regression line accounts for 87.26% of the variance in the data.
- Suppose we rescale the data:

log - log rescale

⇒

allometric

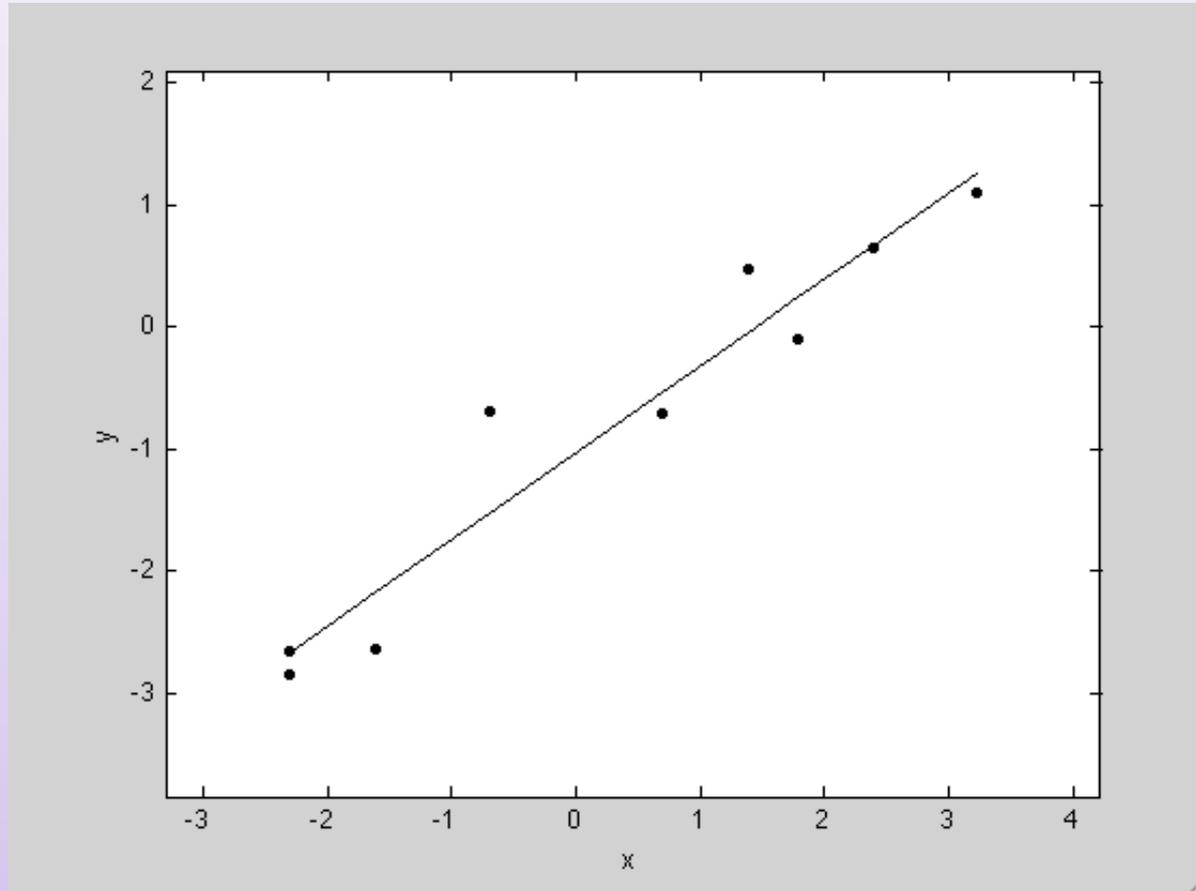
$\ln x$	$\ln y$
-2.3	-2.659
-2.3	-2.847
-1.6	-2.645
-0.7	-0.693
0.7	-0.713
1.4	0.470
1.8	-0.105
2.4	0.642
3.2	1.099



4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Example 4.11 (Mutation Rates)

- A scatterplot of the data with LSR:



4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Example 4.11 (Mutation Rates)

- Is this a good fit?
- The MATLAB output:
 - Eqn for LSR: $\ln y = 0.709705 \ln x + -1.031581$
 - $\rho = 0.962501$
 - The regression line accounts for 92.64% of the variance in the data
- Which model should we choose?

4. (4.4) Rescaling data: Log-Log & Semi-Log Graphs

Example 4.11 (Mutation Rates)

- If we choose the allometric model, we need to recover x and y :

$$\ln y = 0.7097 \ln x - 1.0316$$

$$e^{\ln y} = e^{0.7097 \ln x - 1.0316}$$

$$y = e^{\ln x^{0.7097}} e^{-1.0316}$$

$$y = x^{0.7097} e^{-1.0316}$$

$$y = 0.3564 x^{0.7097}$$

- Now we can use the model to predict. Suppose we know a certain species has a generation time of 10 years, we could interpolate the genomic mutation rate of this species:

$$y = 0.3564(10)^{0.7097} \approx 1.8268 \text{ mutations per generation}$$

Homework

Chapter 4: 4.1 b,d,e,g,h 4.2, 4.3, 4.5, 4.7, 4.8, 4.10, 4.11 4.12

some answers:

4.5 a. $y = 100 (1.26)^x$ b. 5.9 months

4.7 a. $L = (10)^{-2/3} R^{2/3}$ b. plant A has $2^{2/3}$ more leaf biomass than plant B