Chapter 5: Sequences & Discrete Difference Equations

1. (5.1) Sequences
2. (5.2) Limit of a Sequence
3. (5.3) Discrete Difference Equations
4. (5.4) Geometric & Arithmetic Sequences
5. (5.5) Linear Difference Equation with Constant Coefficients (scanned notes)

Sequences

- Recall from Chapter 3 that bivariate data are often displayed as ordered pairs (x_1, y_1), (x_2, y_2), …, (x_n, y_n) or in a table:

x	x_1	x_2	…	x_n
y	y_1	y_2	…	y_n

- A *sequence* is simply a particular kind of bivariate data set:

x	1	2	…	n
y	y_1	y_2	…	y_n

- Or sometimes:

x	0	1	…	n-1
y	y_0	y_1	…	y_{n-1}

Example 5.1

- Consider the following bivariate data set reflecting the total count of Northern Cardinals sighted in Tennessee at Christmastime:

Year	Count
1959	2206
1960	2297
1961	2650
1962	2277
1963	2242
1964	2213
1965	2567
1966	3152
1967	2186
1968	2998
1969	2628
1970	3450
1971	2829

Year	Count
1972	3696
1973	4989
1974	3779
1975	4552
1976	3872
1977	4049
1978	4037
1979	3475
1980	4448
1981	3660
1982	5141
1983	4890
1984	3500

Year	Count
1985	5359
1986	4321
1987	5044
1988	3092
1989	5388
1990	4079
1991	4416
1992	4828
1993	4291
1994	4861
1995	4662
1996	4827
1997	4377

Year	Count
1998	5439
1999	4367
2000	6045
2001	4632
2002	6974
2003	4528
2004	6875
2005	5154
2006	6631
2007	7051
2008	4882
2009	6896
2010	6190
2011	6739

- If we think of the year data as "years starting with 1959", then we have the following sequence:

Example 5.1

yrs start 1959	1	2	3	4	5	6	7	8
# birds	2206	2297	2650	2277	2242	2213	2567	3152

yrs start 1959	9	10	11	12	…	51	52	53
# birds	2186	2998	2628	3450	…	6896	6190	6739

Example 5.1

You may think of a sequence as simply an *ordered* list of numbers. That is, even though a sequence is a bivariate data set, the first member of each ordered pair is really just a placeholder:

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n), \ldots$$

$$(1, y_1), (2, y_2), (3, y_3), \ldots, (n, y_n), \ldots$$

$$(y_1)_1, (y_2)_2, (y_3)_3, \ldots, (y_n)_n, \ldots$$

$$y_1, y_2, y_3, \ldots, y_n, \ldots$$

The n^{th} term of the sequence.

Example 5.1

So then, as an ordered list, our previous data set looks like this:

(2206, 2297, 2650, 2277, 2242, 2213, 2567, 3152, 2186, 2998, 2628, 3450, 2829, 3696, 4989, 3779, 4552, 3872, 4049, 4037, 3475, 4448, 3660, 5141, 4890, 3500, 5359, 4321, 5044, 3092, 5388, 4079, 4416, 4828, 4291, 4861, 4662, 4827, 4377, 5439, 4367, 6045, 4632, 6974, 4528, 6875, 5154, 6631, 7051, 4882, 6896, 6190, 6739)

We don't need to list the years explicitly since that information is "contained" implicitly in the ordering of the list.

We can find, for example, the number of cardinals seen in 1969 by finding the 11^{th} term of the above sequence since 1969 is the 11^{th} year starting with 1959. (2628)

Although this list is ordered, technically speaking, however, this list is not a sequence since it has only 53 terms. A sequence should have infinitely many terms.

Example 5.1

Let's pretend for the moment that this (ordered) list does go on indefinitely. Can you tell what the 125th term is?

No. Since these are actual data measurements, there is no way to know in advance how many cardinals will be seen 2083.

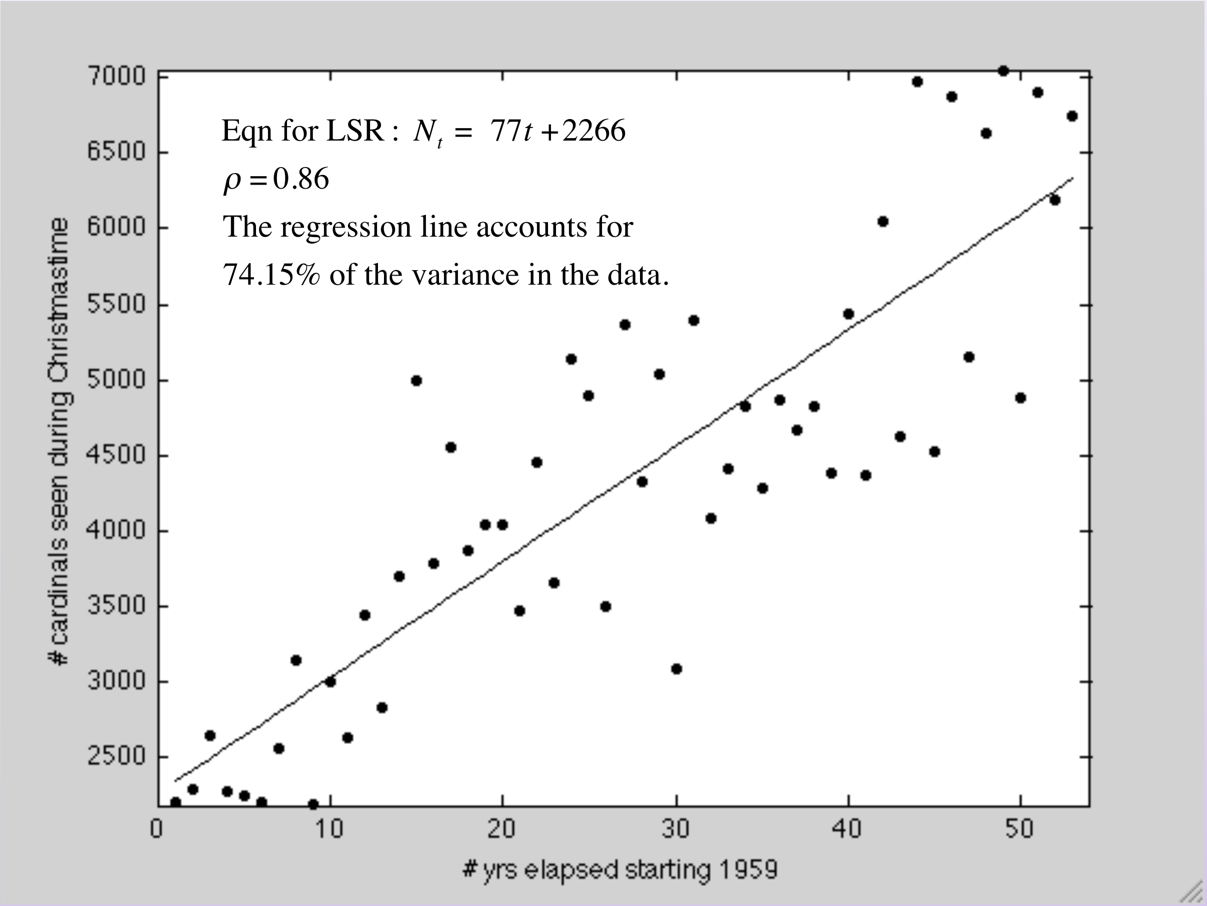
If we build a model for this data, however, we would have a *formula* to determine the forecasted number of cardinals seen in year 2083.

This number would be the 125th term of a *different* sequence- namely, the sequence determined by the model.

Let's use the skills from Unit 1 to find a least squares regression for this data.

Example 5.1

Using our MATLAB program, we have:



Example 5.1

That is, we have a formula that determines a sequence. The number of cardinals N_t seen at Christmastime *t* years elapsed beginning in 1959 is forecast to be given by:

$$N_t = 77t + 2266$$

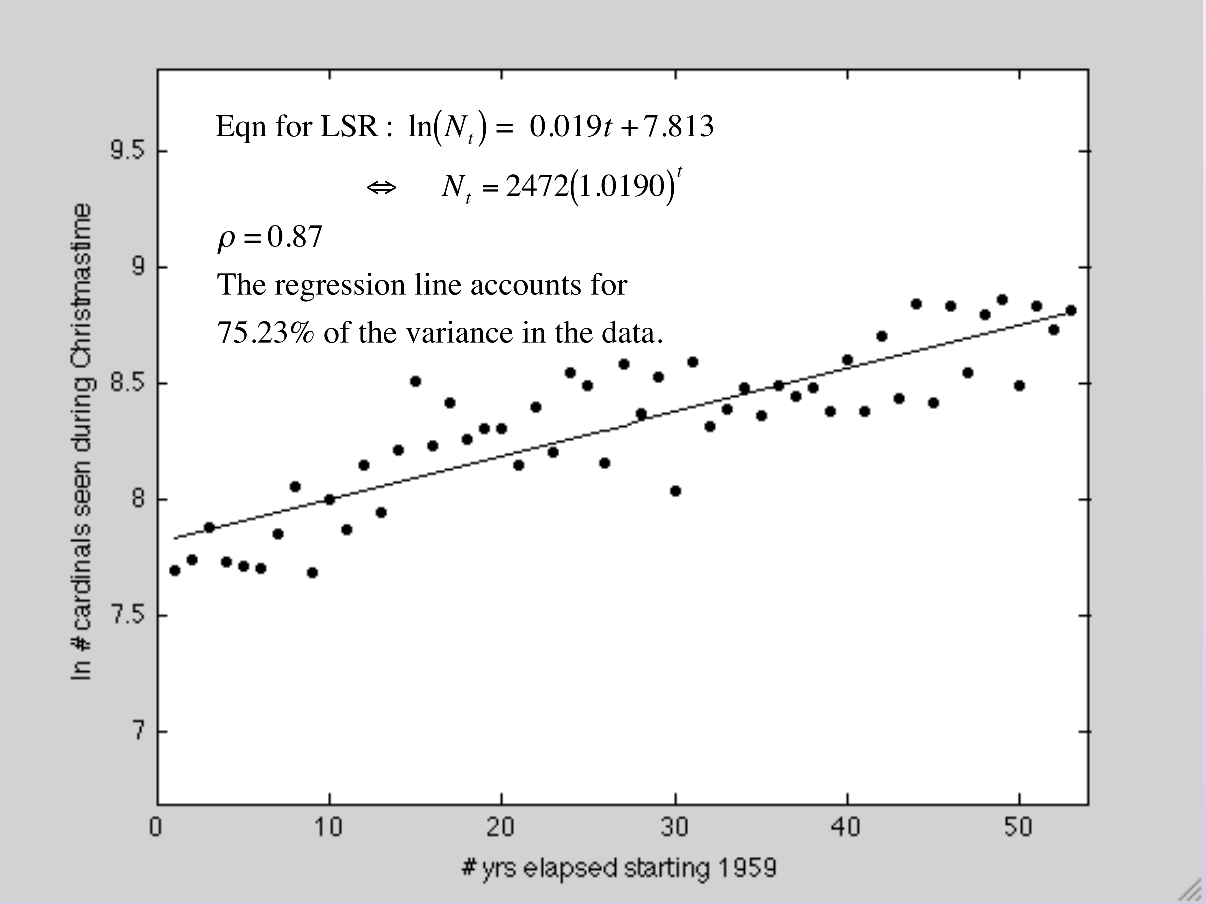
Again, this is not the "sequence" of the data but, rather, a LSR for the data. Interpolating for t=11, we get $N(11)$=3113. Notice this is different from our 11th data point, 2628.

But equipped with a formula that determines our sequence, we can extrapolate to find to the 125th term of our sequence:

$$N_{125} = 77 \cdot 125 + 2266 = 11891$$

To reinforce prior work: sometimes it is reasonable to assume a population is growing exponentially, so let's rescale our data and see what we get:

Example 5.1


Eqn for LSR : $\ln(N_t) = 0.019t + 7.813$
$\Leftrightarrow \quad N_t = 2472(1.0190)^t$
$\rho = 0.87$
The regression line accounts for
75.23% of the variance in the data.
ln # cardinals seen during Christmastime
yrs elapsed starting 1959
9.5
9
8.5
8
7.5
7
0
10
20
30
40
50

Example 5.1

Once again, we have a formula that determines a sequence. The number of cardinals N_t seen at Christmastime *t* years elapsed beginning in 1959 is forecast to be given by:

$$N_t = 2471 \cdot (1.019)^t$$

Again, this is not the “sequence” of the data but, rather, a LSR for the data. Interpolating for *t*=11, we get *N(*11*)*=3039. Notice this is different from our 11th data point, 2628.

But equipped with a formula that determines our sequence, we can extrapolate to find to the 125th term of our sequence:

$$N_{125} = 2471 \cdot (1.019)^{125} = 25980$$

Example 5.2

Consider the sequence given by the formula:

$$a_n = f(n) = (-1)^n \cdot \frac{2n}{n+1}$$

Find the first 5 terms of this sequence.

Solution:

$$a_1 = f(1) = (-1)^1 \cdot \frac{2 \cdot 1}{1+1} = -1$$

$$a_2 = f(2) = (-1)^2 \cdot \frac{2 \cdot 2}{2+1} = \frac{4}{3}$$

$$a_3 = f(3) = (-1)^3 \cdot \frac{2 \cdot 3}{3+1} = -\frac{6}{4} = -\frac{3}{2}$$

$$a_4 = f(4) = (-1)^4 \cdot \frac{2 \cdot 4}{4+1} = \frac{8}{5}$$

$$a_5 = f(5) = (-1)^5 \cdot \frac{2 \cdot 5}{5+1} = -\frac{10}{6} = -\frac{5}{3}$$

The formula in the previous example is an *explicit* formula in the following sense- if you want to know the 125th term of the sequence, you simply “plug in” 125 for n:

$$a_{125} = f(125) = (-1)^{125} \cdot \frac{2 \cdot 125}{125+1} = -\frac{250}{126}$$

More common, however, when building models, we work with a recurrence formula or recurrence relation.

For example, consider a population that doubles each year. If we let x_n represent the size of the population at time step n, then we can model how this population changes from one time step to the next by the equation:

$$x_{n+1} = 2x_n$$

How is this different? Well, let's consider how we would find the population size after 125 time steps:

$$x_{n+1} = 2x_n \quad \Rightarrow \quad x_{125} = 2x_{124}$$

$$\Leftrightarrow \quad x_{125} = 2(2x_{123})$$

$$\Leftrightarrow \quad x_{125} = 2(2(2x_{122}))$$

$$\vdots$$

$$\Leftrightarrow \quad x_{125} = ?$$

So, in some sense, to find the 125th term, we need to know all of the previous terms. This is very different from the previous example.

Fibonacci Sequence

A famous example of a sequence generated by a recurrence relation is the Fibonacci sequence. Consider a population of rabbits. If we let $x_0=1$ and $x_1=1$, then the population size of the n^{th} generation of rabbits can be modeled by the recurrence relation:

$$x_{n+1} = x_n + x_{n-1}$$

Let's generate some terms of the associated sequence:

$$x_2 = x_1 + x_0 = 1 + 1 = 2 \qquad x_5 = x_4 + x_3 = 5 + 3 = 8$$

$$x_3 = x_2 + x_1 = 2 + 1 = 3 \qquad x_6 = x_5 + x_4 = 8 + 5 = 13$$

$$x_4 = x_3 + x_2 = 3 + 2 = 5 \qquad x_7 = x_6 + x_5 = 13 + 8 = 21$$

We have: 1,1,2,3,5,8,13,21,? 34,55,89,144,…

Difference Equations

In general, suppose we have a quantity- like a population- whose value at time step n+1 depends on the values at each of the previous time steps. That is,

$$x_{n+1} = f\left(x_n, x_{n-1}, \ldots, x_0\right)$$

An equation that can be written in this form is called a **difference equation**. If the value at step n+1 depends only on the value at the previous step, that is, if:

$$x_{n+1} = f\left(x_n\right) \qquad \text{example}: \; x_{n+1} = 2x_n$$

then it's a **first order difference equation**.

If the value at step n+1 depends on the values at the two previous steps, that is, if:

$$x_{n+1} = f\left(x_n, x_{n-1}\right) \quad \text{example}: \; x_{n+1} = x_n + x_{n-1}$$

then it's a **second order difference equation**.

As mentioned above, to find, say, the 125th term, we would need to know all of the previous terms:

$$x_{n+1} = 2x_n \quad \Rightarrow \quad x_{125} = 2x_{124}$$

$$\Leftrightarrow \quad x_{125} = 2(2x_{123})$$

$$\vdots$$

$$\Leftrightarrow \quad x_{125} = ?$$

Unless, that is, we can find an explicit formula for the nth term that does not depend on any of the previous terms. In other words, we'd like to replace our recurrence formula with an "explicit" one:

$$x_{n+1} = f(x_n) \quad \rightarrow \quad x_n = f(n)$$

Example 5.4

A population of doves increases by 3% each year. Let x_n be the size of the population at year n. Then:

$$\begin{aligned} x_{n+1} &= f(x_n) \\ &= x_n + 0.03x_n \\ &= 1.03x_n \end{aligned}$$

Let x_0 be the initial population size. Then we have:

$$\begin{aligned} x_1 &= 1.03x_0 \\ x_2 &= 1.03x_1 = 1.03(1.03x_0) = 1.03^2 x_0 \\ x_3 &= 1.03x_2 = 1.03(1.03^2 x_0) = 1.03^3 x_0 \\ &\vdots \\ x_n &= 1.03^n x_0 \end{aligned}$$

Geometric Sequences

The example we just looked at was an example of a geometric sequence. A **geometric sequence** is a sequence with the form: $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$

where a and r are numbers.

Notice that this sequence is generated by the form of that generic term. And the generic term, in this case, was found by solving the first order difference equation:

$$a_{n+1} = r \cdot a_n, \text{ where } a_0 = a$$

$$\Rightarrow \quad a_1 = r \cdot a_0 = r \cdot a$$

$$\Leftrightarrow \quad a_2 = r \cdot a_1 = r \cdot (r \cdot a) = r^2 a$$

$$\Leftrightarrow \quad a_n = r^n \cdot a_0 = ar^n$$

Example 5.5 (Wild Hares)

A population of wild hares increases by 13% each year. Currently, there are 200 hares. If x_n is the number of hares in the population at the end of year n, find:

(a) the difference equation relating x_{n+1} to x_n

Solution: Since the population increases by 13% each year, the difference equation is:

$$x_{n+1} = 1.13x_n$$

(b) the general solution to the difference equation found in part a. Solution: $x_n = 1.13^n x_0 = 1.13^n(200)$

(c) the number of hares in the population at the end of six years from now. Solution: $x_6 = 1.13^6(200) \approx 416$

Thus, at the end of year six there are approximately 416 hares.

Arithmetic Sequences

Another common sequence is an arithmetic sequence. An **arithmetic sequence** is a sequence with the form:

$$a, a+d, a+2d, a+3d, \ldots, \boxed{a+nd}, \ldots$$

where a and d are numbers.

Notice that this sequence is generated by the form of that generic term. And the generic term, in this case, is found by solving the first order difference equation:

$$a_{n+1} = a_n + d, \text{ where } a_0 = a$$

$$\Rightarrow \quad a_1 = a_0 + d = a + d$$

$$\Leftrightarrow \quad a_2 = a_1 + d = (a+d) + d = a + 2d$$

$$\vdots$$

$$\Leftrightarrow \quad a_n = a + nd$$

Homework

Chapter 5:

5.2, 5.3, 5.5, 5.8, 5.9

Some answers:

5.5 (a) $x_{n+1}=1.1x_n$ (b) $x_n=50(1.1)^n$

5.8 (a) $x_n=800(1.1)^n + 200$ (b) no (c) 68

5.9 (a) 5 (b) extinction