Chapter 7: Matrix Algebra

1. (7.1) Matrix Arithmetic
   a) Matrix-Vector Multiplication
   b) Matrix-Matrix Multiplication
2. (7.2) Applications

Motivation

- Recall once again Example 6.4:
- Initially our wetland is completely submerged and we know that every 10 years:
  - 5% of submerged wetlands become saturated wetland
  - 12% of saturated wetlands become dry
  - 100% of dry wetlands remain dry
- We found the dynamic equations describing the change in composition each time step:

$$u(t+1) = 0.95u(t) + 0.00s(t) + 0.00d(t)$$
$$s(t+1) = 0.05u(t) + 0.88s(t) + 0.00d(t)$$
$$d(t+1) = 0.00u(t) + 0.12s(t) + 1.00d(t)$$

Motivation

- These equations are written succinctly as:

$$\begin{bmatrix} u(t+1) \\ s(t+1) \\ d(t+1) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix} \text{ or } \mathbf{v}(t+1) = A \cdot \mathbf{v}(t)$$

- This last equation needs to be justified; that is, does matrix-vector multiplication accomplish here what we need?
- In some sense, this is the wrong question- we should *define* some operation that does what we want; it just so turns out that the operation we want is what we call matrix multiplication.

Motivation

- So, let's look once again at the dynamic equations and describe the operation that will do the job:

$$u(t+1) = 0.95u(t) + 0.00s(t) + 0.00d(t)$$
$$s(t+1) = 0.05u(t) + 0.88s(t) + 0.00d(t)$$
$$d(t+1) = 0.00u(t) + 0.12s(t) + 1.00d(t)$$

Motivation

- So, let's look once again at the dynamic equations and describe the operation that will do the job:

$$u(t+1) = 0.95u(t) + 0.00s(t) + 0.00d(t)$$
$$s(t+1) = 0.05u(t) + 0.88s(t) + 0.00d(t)$$
$$d(t+1) = 0.00u(t) + 0.12s(t) + 1.00d(t)$$

- Notice that we will need each entry of the first column of our transfer matrix to be multiplied by $u(t)$.

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix}$$

Motivation

- So, let's look once again at the dynamic equations and describe the operation that will do the job:

$$u(t+1) = 0.95u(t) + 0.00s(t) + 0.00d(t)$$
$$s(t+1) = 0.05u(t) + 0.88s(t) + 0.00d(t)$$
$$d(t+1) = 0.00u(t) + 0.12s(t) + 1.00d(t)$$

- And we need each entry of the second column of our transfer matrix to be multiplied by $s(t)$.

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix}$$

Motivation

- So, let's look once again at the dynamic equations and describe the operation that will do the job:

$$u(t+1) = 0.95u(t) + 0.00s(t) + \boxed{0.00}\,d(t)$$
$$s(t+1) = 0.05u(t) + 0.88s(t) + \boxed{0.00}\,d(t)$$
$$d(t+1) = 0.00u(t) + 0.12s(t) + \boxed{1.00}\,d(t)$$

- Finally, we need each entry of the third column of our transfer matrix to be multiplied by $d(t)$.

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix}$$

Motivation

- So, we simply decide to let this:

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix}$$

- mean this:

$$\begin{bmatrix} 0.95u(t) + 0.00s(t) + 0.00d(t) \\ 0.05u(t) + 0.88s(t) + 0.00d(t) \\ 0.00u(t) + 0.12s(t) + 1.00d(t) \end{bmatrix}$$

Motivation

- And we'll have what we want:

$$\begin{bmatrix} u(t+1) \\ s(t+1) \\ d(t+1) \end{bmatrix} = \begin{bmatrix} 0.95u(t) + 0.00s(t) + 0.00d(t) \\ 0.05u(t) + 0.88s(t) + 0.00d(t) \\ 0.00u(t) + 0.12s(t) + 1.00d(t) \end{bmatrix}$$

- But this is precisely the way that matrix-vector multiplication is defined!
- If we look a bit closer, we can see two different points of view, or two different ways to think about this operation:

Motivation

- One way to think about this operation is as we just did-column-wise:

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix} = \begin{bmatrix} 0.95u(t) + \\ 0.05u(t) + \\ 0.00u(t) + \end{bmatrix}$$

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix} = \begin{bmatrix} 0.95u(t) + 0.00s(t) + \\ 0.05u(t) + 0.88s(t) + \\ 0.00u(t) + 0.12s(t) + \end{bmatrix}$$

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix} = \begin{bmatrix} 0.95u(t) + 0.00s(t) + 0.00d(t) \\ 0.05u(t) + 0.88s(t) + 0.00d(t) \\ 0.00u(t) + 0.12s(t) + 1.00d(t) \end{bmatrix}$$

Motivation

- Another way to think about this operation is as rows acting on the column vector:

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix} = \begin{bmatrix} 0.95u(t) + 0.00s(t) + 0.00d(t) \\ \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix} = \begin{bmatrix} 0.95u(t) + 0.00s(t) + 0.00d(t) \\ 0.05u(t) + 0.88s(t) + 0.00d(t) \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix} = \begin{bmatrix} 0.95u(t) + 0.00s(t) + 0.00d(t) \\ 0.05u(t) + 0.88s(t) + 0.00d(t) \\ 0.00u(t) + 0.12s(t) + 1.00d(t) \end{bmatrix}$$

Motivation

- Both points of view yield the same result:

$$\begin{bmatrix} 0.95u(t) + 0.00s(t) + 0.00d(t) \\ 0.05u(t) + 0.88s(t) + 0.00d(t) \\ 0.00u(t) + 0.12s(t) + 1.00d(t) \end{bmatrix}$$

- Different contexts will dictate which point of view one chooses to adopt.
- Matrix-vector multiplication is a special case of matrix-matrix multiplication
  - We will soon need to use this more general operation
  - So, we now undertake a review of basic Matrix Arithmetic

- Matrix Addition/Subtraction
  - This operation is defined just as we did for vectors; that is, addition and subtraction for matrices is **componentwise**
  - As before, this operation makes no sense for matrices with different dimensions
- Example 7.1: A and M are the 2x3 matrices shown below. Add them.

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 6 & 3 & 7 \end{bmatrix}; \quad M = \begin{bmatrix} 2 & 5 & 5 \\ 9 & 1 & 7 \end{bmatrix}$$

solution:

$$A + M = \begin{bmatrix} 4 & 9 & 13 \\ 15 & 4 & 14 \end{bmatrix}$$

- Matrix-Scalar Multiplication
  - This operation is also defined just as we did for vectors; that is, matrix-scalar multiplication is **componentwise**
- Example:

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 6 & 3 & 7 \end{bmatrix}; \quad c = -2$$

$$cA = \begin{bmatrix} -4 & -8 & -16 \\ -12 & -6 & -14 \end{bmatrix}$$

Matrix Multiplication

- Matrix Multiplication is **NOT** what one might think!
  - It is NOT componentwise; let A and M be as before, then:

$$A \cdot M = \begin{bmatrix} 2 & 4 & 8 \\ 6 & 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 & 5 \\ 9 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 20 & 40 \\ 54 & 3 & 49 \end{bmatrix}$$

  - Moreover, dimension compatibility for matrix multiplication is NOT (necessarily) having the same dimensions; for instance:

$$A \cdot M = \begin{bmatrix} 2 & 4 & 8 \\ 6 & 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 & 5 \\ 9 & 1 & 7 \end{bmatrix} = something?$$

  - Finally, matrix multiplication is NOT commutative:

$$A \cdot M = M \cdot A$$

Matrix Multiplication

- Before giving the definition of matrix multiplication, let's first see an example where it is defined and how it is performed
- Example 7.2: Let A and B be the following matrices. Find AB.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$A \cdot B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 0\cdot 2 + 2\cdot 1 & 1\cdot 1 + 0\cdot 1 + 2\cdot 0 \\ -1\cdot 3 + 3\cdot 2 + 1\cdot 1 & -1\cdot 1 + 3\cdot 1 + 1\cdot 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

Note: (2x3) (3x2) = (2x2)

Note: In this case BA is defined (it will be a 3x3 matrix and it is NOT the same as AB), but this is NOT always the case!

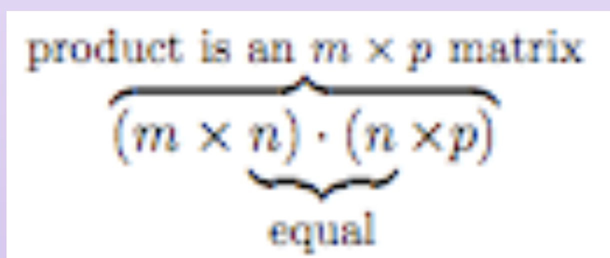
1. (7.1) Matrix Arithmetic

Matrix Multiplication

- Definition: If A is an m × n matrix and B is an n × p matrix, then their product is an m × p matrix denoted by AB. If $<AB>_{ij}$ denotes the i,j entry of the product, then:

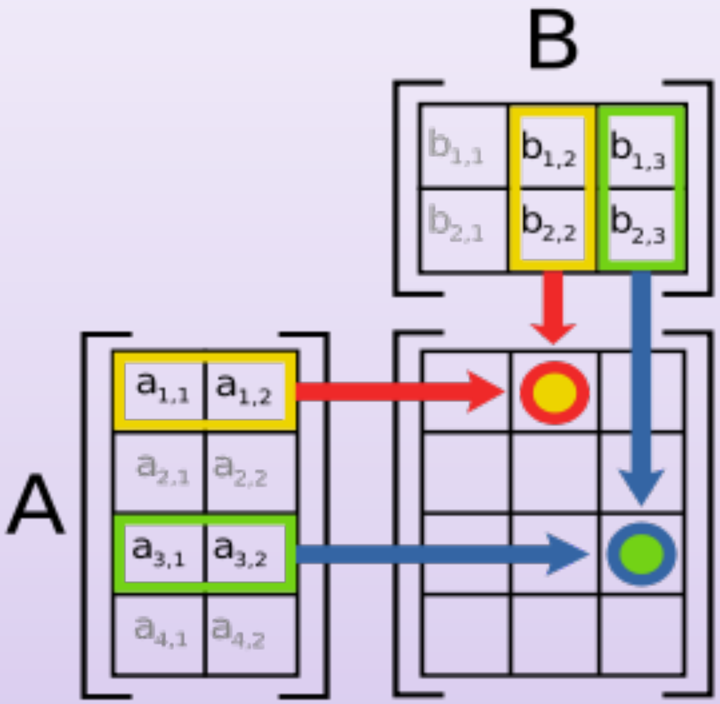
$$\langle AB \rangle_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

- Note the dimension compatibility conditions for matrix multiplication:



Matrix Multiplication

- Heuristically, let A be 4x2 and B be 2x3. Then



Matrix Multiplication

- Let's revisit example 7.2: Let A and B be the following matrices. This time, instead of finding AB, we calculate BA:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$B \cdot A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 1(-1) & 3\cdot 0 + 1\cdot 3 & 3\cdot 2 + 1\cdot 1 \\ 2\cdot 1 + 1(-1) & 2\cdot 0 + 1\cdot 3 & 2\cdot 2 + 1\cdot 1 \\ 1\cdot 1 + 0(-1) & 1\cdot 0 + 0\cdot 3 & 1\cdot 2 + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$$

Note: (3x2) (2x3) = (3x3)

Note: In this case BA is defined, but it is obviously not equal to AB.

1. (7.1) Matrix Arithmetic

Matrix Multiplication

- Example 7.3:

  Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Does the product Ax exist?

  If so what is it? Does the product xA exist? If so what is it?
- Solution: Notice that x is a vector. We can think of a vector as a matrix that only has one column. So, x is a 2x1 matrix. The process for multiplication remains the same.
- First we check if the product Ax is possible under matrix multiplication:

$$\overbrace{(2 \times \underbrace{2) \cdot (2}_{\text{equal}} \times 1)}^{\text{product is an } 2 \times 1 \text{ matrix}}$$

Matrix Multiplication

- So the product is defined and we have:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 0\cdot(-1) \\ 2\cdot 3 + 1\cdot(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

- Now, we check if the product xA is possible under matrix multiplication:

$$(2 \times \underbrace{1) \cdot (2}_{\text{not equal}} \times 2).$$

Thus, the matrix multiplication xA is not defined.

Example 7.4.1 (Example 6.4 version 3.0)

- Recall our simple model for ecological succession of a coastal wetlands:

$$\begin{bmatrix} u(t+1) \\ s(t+1) \\ d(t+1) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ s(t) \\ d(t) \end{bmatrix} \quad \text{or} \quad \mathbf{v}(t+1) = A \cdot \mathbf{v}(t)$$

- We are now in a position to justify that the above matrix equation does what we want it to do. We found that if

$$\mathbf{v}(0) = \begin{bmatrix} u(0) \\ s(0) \\ d(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ then } \quad \mathbf{v}(1) = \begin{bmatrix} 0.95 \\ 0.05 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}(2) = \begin{bmatrix} 0.9025 \\ 0.0915 \\ 0.006 \end{bmatrix}.$$

Example 7.4.1 (Example 6.4 version 3.0)

- We now calculate these using matrix-vector multiplication:

$$\begin{bmatrix} u(1) \\ s(1) \\ d(1) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} u(0) \\ s(0) \\ d(0) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.95 \\ 0.05 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u(2) \\ s(2) \\ d(2) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ s(1) \\ d(1) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} 0.95 \\ 0.05 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.9025 \\ 0.0915 \\ 0.0060 \end{bmatrix}$$

- While this is a marked improvement over our previous method, it is still not ideal; for example, what if I would like to know the wetland composition after 12 decades?

2. (7.2) Applications

- To find the wetland composition after 12 decades, the present method requires us to use the vector for the wetland composition after 11 decades:

$$\begin{bmatrix} u(12) \\ s(12) \\ d(12) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} u(11) \\ s(11) \\ d(11) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} ??? \\ ??? \\ ??? \end{bmatrix}$$

- But, to find the wetland composition after 11 decades, we'll need to find the vector for the wetland composition after 10 decades:

$$\begin{bmatrix} u(11) \\ s(11) \\ d(11) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} u(10) \\ s(10) \\ d(10) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} ??? \\ ??? \\ ??? \end{bmatrix}$$

- It’s easy to see where this is going- if we want to find the wetland composition after, say, t decades, the present method requires us to know the vectors for the wetland composition for all previous decades back to the initial composition vector
- Let’s see if we can find an even better method:

$$\mathbf{v}(1) = A \cdot \mathbf{v}(0)$$
$$\mathbf{v}(2) = A \cdot \mathbf{v}(1) = A \cdot A \cdot \mathbf{v}(0) \stackrel{?}{=} A^2 \cdot \mathbf{v}(0)$$
$$\mathbf{v}(3) = A \cdot \mathbf{v}(2) = A \cdot A^2 \cdot \mathbf{v}(0) = A^3 \cdot \mathbf{v}(0)$$
$$\vdots$$
$$\mathbf{v}(t) = A^t \cdot \mathbf{v}(0)$$

Example 7.4.2 (Example 6.4 version 4.0)

- The questioned equality on the previous slide is, indeed, true; that is, matrix-matrix multiplication does what we want it to do
- We illustrate this by, once again, finding the wetland composition vector after 2 decades:

$$\begin{bmatrix} u(2) \\ s(2) \\ d(2) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix}^2 \begin{bmatrix} u(0) \\ s(0) \\ d(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example 7.4.2 (Example 6.4 version 4.0)

- We'll calculate that product step by step for practice:

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} = \begin{bmatrix} 0.9025 & 0 & 0 \\ & & \\ & & \end{bmatrix}$$

$$(0.95)(0.95) + (0)(0.05) + (0)(0)$$

$$(0.95)(0) + (0)(0.88) + (0)(0.12)$$

$$(0.95)(0) + (0)(0) + (0)(1)$$

Example 7.4.2 (Example 6.4 version 4.0)

- We'll calculate that product step by step for practice:

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} = \begin{bmatrix} 0.9025 & 0 & 0 \\ 0.0915 & 0.7744 & 0 \\ & & \end{bmatrix}$$

$$(0.05)(0.95) + (0.88)(0.05) + (0)(0)$$

$$(0.05)(0) + (0.88)(0.88) + (0)(0.12)$$

$$(0.05)(0) + (0.88)(0) + (0)(1)$$

Example 7.4.2 (Example 6.4 version 4.0)

- We'll calculate that product step by step for practice:

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} = \begin{bmatrix} 0.9025 & 0 & 0 \\ 0.0915 & 0.7744 & 0 \\ 0.0060 & 0.2256 & 1 \end{bmatrix}$$

$$(0)(0.95)+(0.12)(0.05)+(1)(0)$$

$$(0)(0)+(0.12)(0.88)+(1)(0.12)$$

$$(0)(0)+(0.12)(0)+(1)(1)$$

Example 7.4.2 (Example 6.4 version 4.0)

- Now back to the problem:

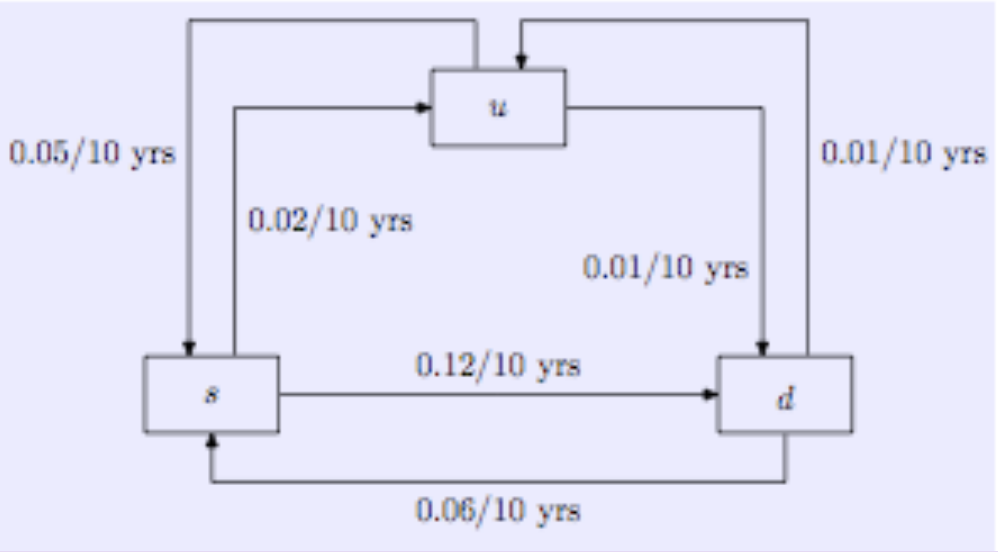
$$\begin{bmatrix} u(2) \\ s(2) \\ d(2) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix}^2 \begin{bmatrix} u(0) \\ s(0) \\ d(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.88 & 0 \\ 0 & 0.12 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9025 & 0 & 0 \\ 0.0915 & 0.7744 & 0 \\ 0.0060 & 0.2256 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.9025 \\ 0.0915 \\ 0.0060 \end{bmatrix}$$ Just as before!

Example 7.5 (Example 6.5 version 2.0)

Recall:



$$\begin{bmatrix} 0.94 & 0.02 & 0.01 \\ 0.05 & 0.86 & 0.06 \\ 0.01 & 0.12 & 0.93 \end{bmatrix}$$

Example 7.5 (Example 6.5 version 2.0)

- We find the wetland composition after 2 decades for this more complex ecological succession model:

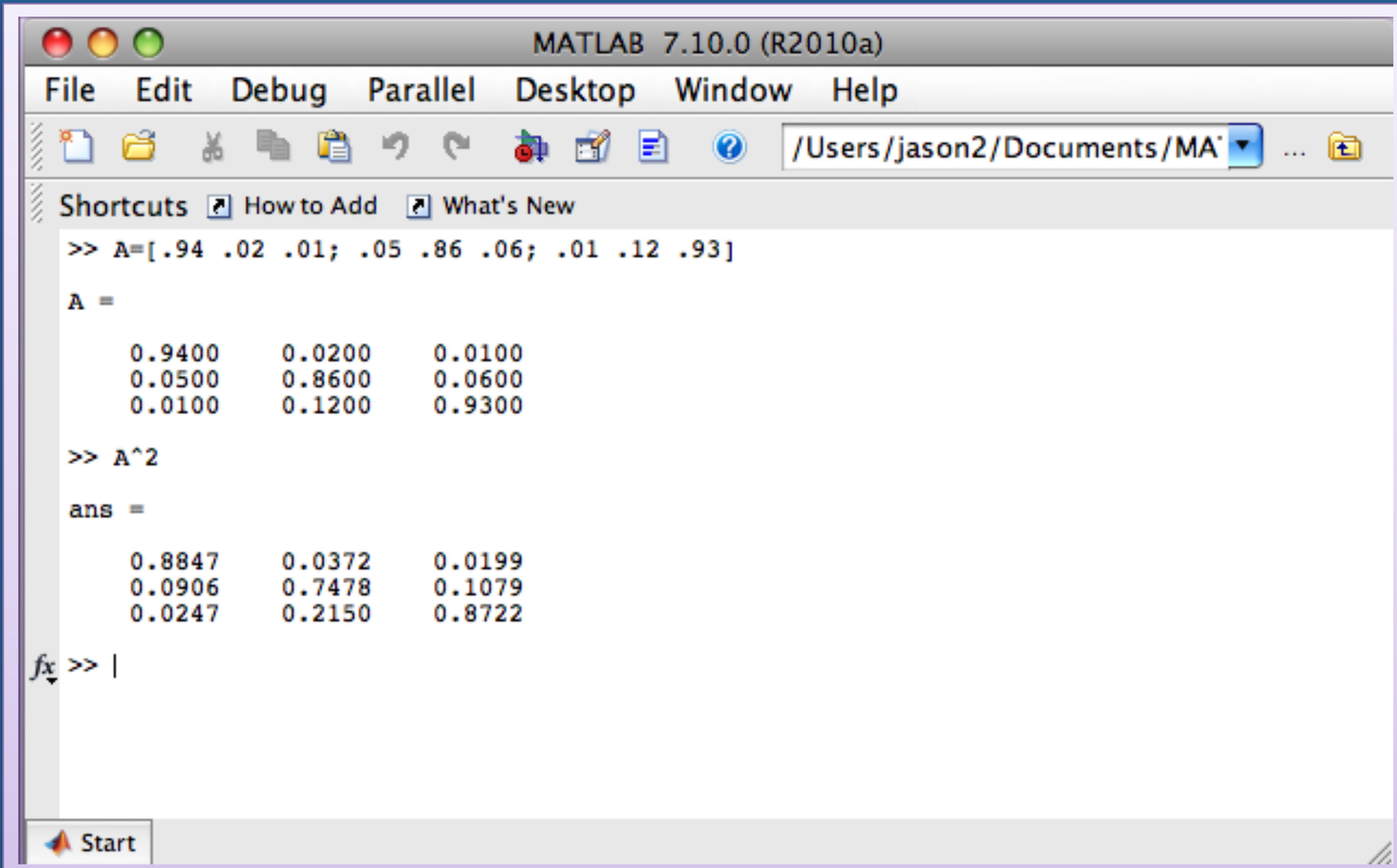
$$\begin{bmatrix} u(2) \\ s(2) \\ d(2) \end{bmatrix} = \begin{bmatrix} 0.94 & 0.02 & 0.01 \\ 0.05 & 0.86 & 0.06 \\ 0.01 & 0.12 & 0.93 \end{bmatrix}^2 \begin{bmatrix} u(0) \\ s(0) \\ d(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0.94 & 0.02 & 0.01 \\ 0.05 & 0.86 & 0.06 \\ 0.01 & 0.12 & 0.93 \end{bmatrix} \cdot \begin{bmatrix} 0.94 & 0.02 & 0.01 \\ 0.05 & 0.86 & 0.06 \\ 0.01 & 0.12 & 0.93 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Let's use MATLAB to do the heavy lifting this time:

2. (7.2) Applications

Example 7.5 (Example 6.5 version 2.0)



Example 7.5 (Example 6.5 version 2.0)

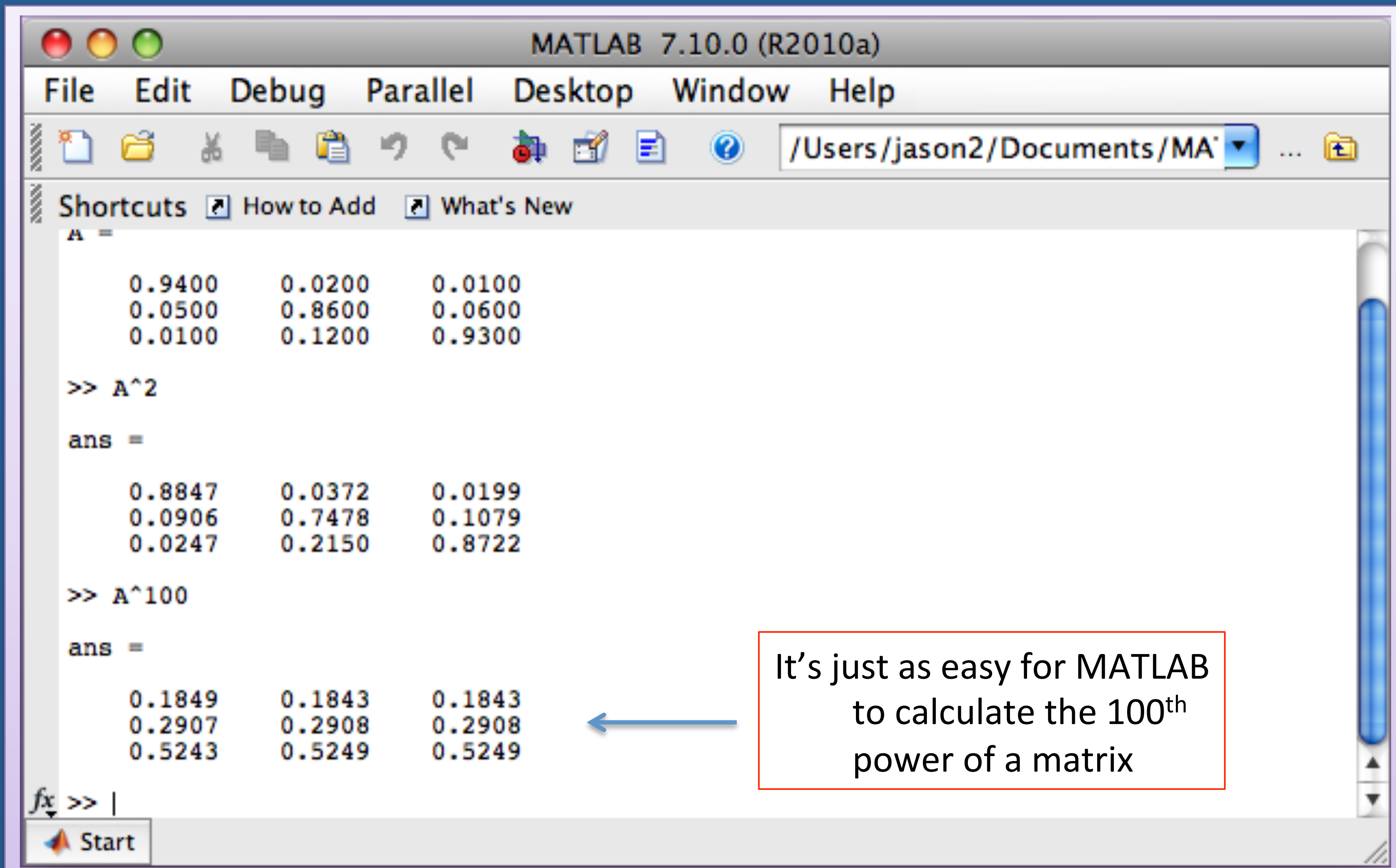
- And we have:

$$\begin{bmatrix} u(2) \\ s(2) \\ d(2) \end{bmatrix} = \begin{bmatrix} 0.94 & 0.02 & 0.01 \\ 0.05 & 0.86 & 0.06 \\ 0.01 & 0.12 & 0.93 \end{bmatrix}^2 \begin{bmatrix} u(0) \\ s(0) \\ d(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0.8847 & 0.0372 & 0.0199 \\ 0.0906 & 0.7478 & 0.1079 \\ 0.0247 & 0.2150 & 0.8722 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8847 \\ 0.0906 \\ 0.0247 \end{bmatrix}$$

Example 7.5 (Example 6.5 version 2.0)



It's just as easy for MATLAB to calculate the 100th power of a matrix

Example 7.5 (Example 6.5 version 2.0)

- Hence:

$$\begin{bmatrix} u(100) \\ s(100) \\ d(100) \end{bmatrix} = \begin{bmatrix} 0.94 & 0.02 & 0.01 \\ 0.05 & 0.86 & 0.06 \\ 0.01 & 0.12 & 0.93 \end{bmatrix}^{100} \begin{bmatrix} u(0) \\ s(0) \\ d(0) \end{bmatrix}$$

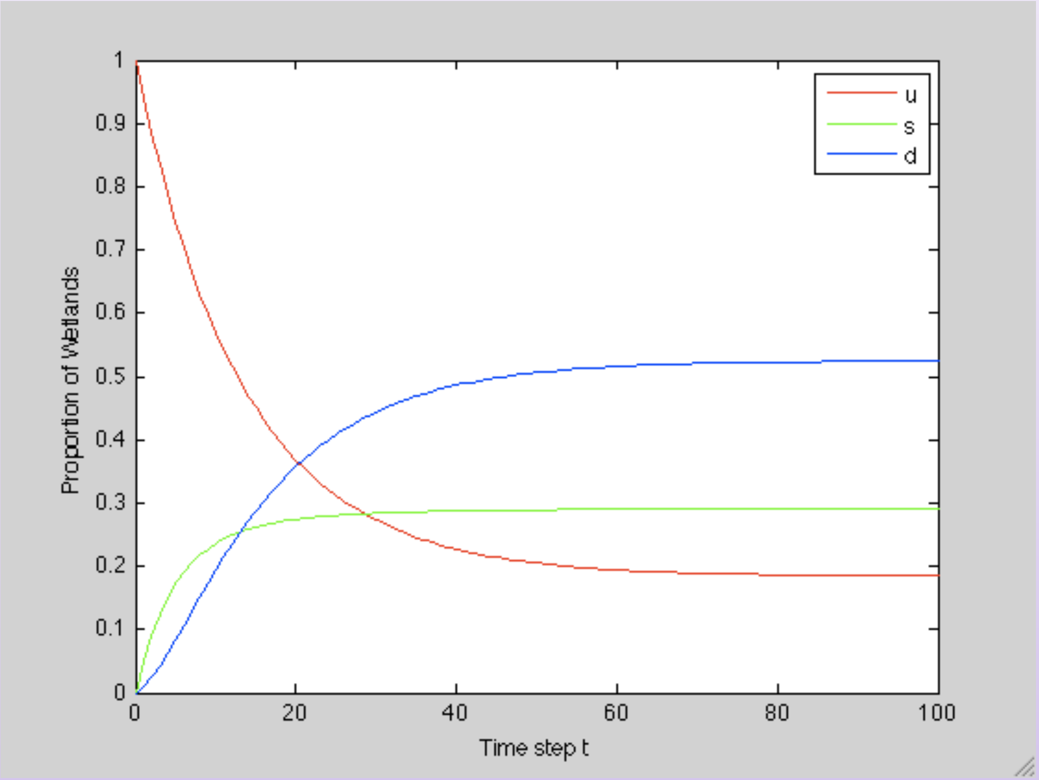
$$= \begin{bmatrix} 0.1849 & 0.1843 & 0.1843 \\ 0.2907 & 0.2908 & 0.2908 \\ 0.5243 & 0.5249 & 0.5249 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1849 \\ 0.2907 \\ 0.5243 \end{bmatrix}$$

What appears to be happening?

Example 7.5 (Example 6.5 version 2.0)

- Next week we will be writing an m-file that plots time series data for an ecological succession model
- For this problem, the output for the first 100 decades is:



Homework

- Chapter 6: 6.2-6.6
- Chapter 7: 7.1-2, 7.4-7.8
- Quiz 3: Covers chapters 5, 6 and 7
  - Find the general solution for one of each of the three cases we considered
  - There will be a problem similar to Exercise 5.8
  - Given a flow diagram, construct the transfer matrix
  - Give an ecological interpretation of each entry of the transfer matrix
  - Construct an initial composition vector from given information
  - Formulate an equation for finding the composition vector after t time steps
  - Calculate (by hand) the composition vector after 1 and after 2 time steps