

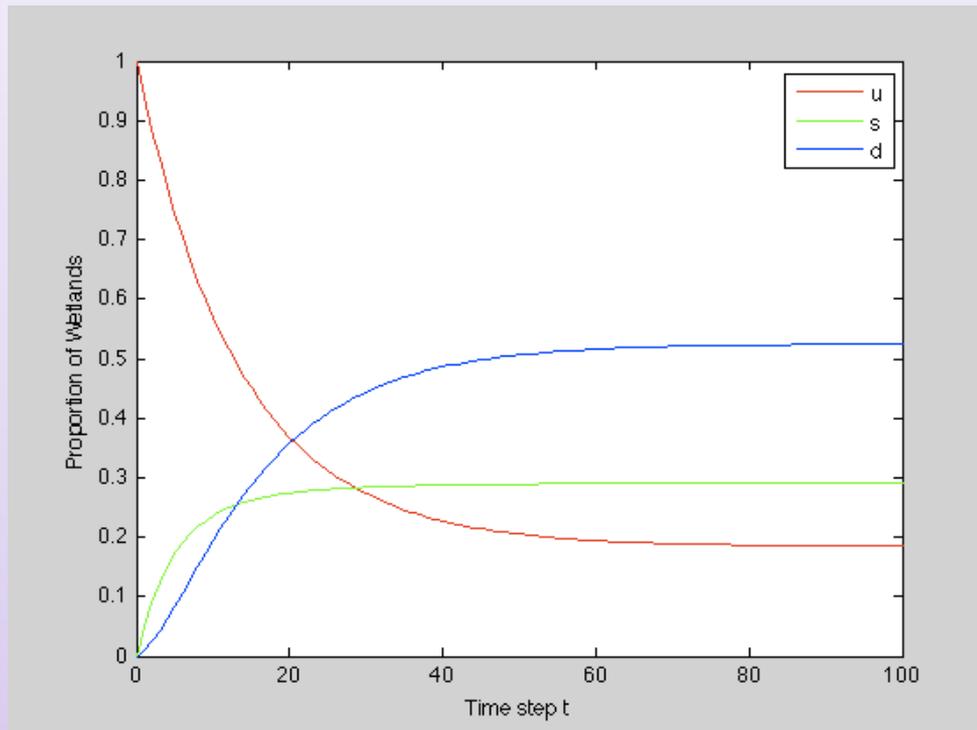
Chapter 8: Long-Term Dynamics or Equilibrium

1. (8.1) Equilibrium
2. (8.2) Eigenvectors
3. (8.3) Stability

1. (8.1) Equilibrium

Notion of a Long-Term Equilibrium State

- Recall from example 7.5 the time series plot showing the wetland composition over the first 100 decades:

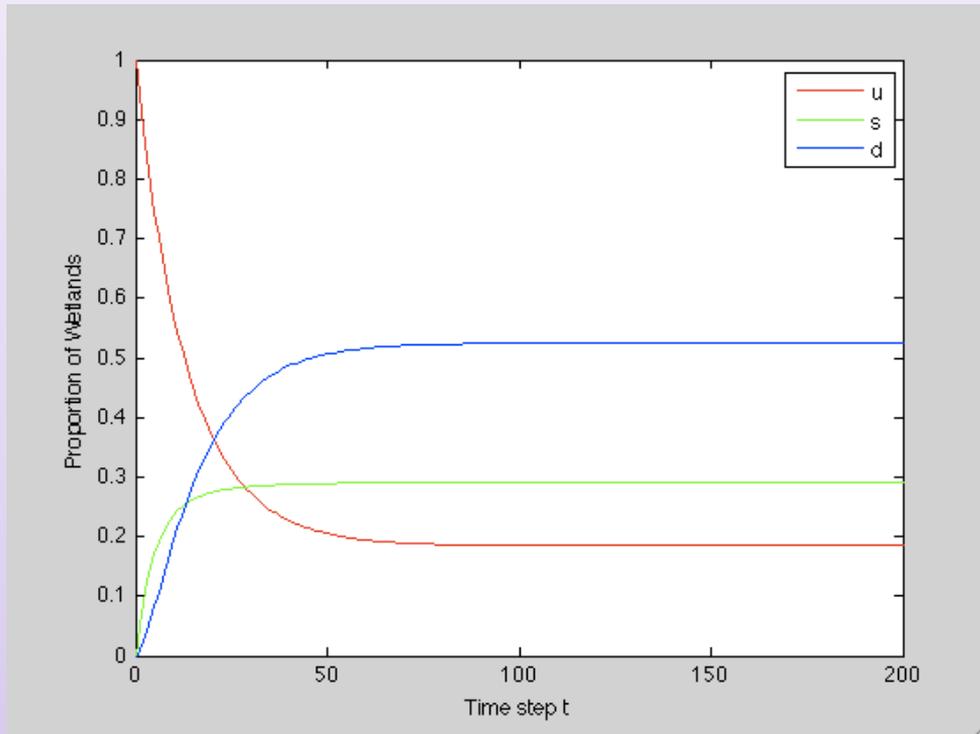


- After some transient behavior, it appears that the wetland composition has settled to some fixed proportions.

1. (8.1) Equilibrium

Notion of a Long-Term Equilibrium State

- To further investigate, let's look at a plot for the first 200 decades:



In fact:

$$\mathbf{v}(100) = \begin{bmatrix} 0.1850 \\ 0.2907 \\ 0.5243 \end{bmatrix}$$

$$\mathbf{v}(200) = \begin{bmatrix} 0.1844 \\ 0.2908 \\ 0.5248 \end{bmatrix}$$

1. (8.1) Equilibrium

Notion of a Long-Term Equilibrium State

- We see that each class of the system approaches some fixed fraction/proportion
- What we want to do now is define a *mathematical* method to find this long-term state
 - It turns out that this long-term steady state is represented by an *eigenvector* of the transfer matrix
 - In the models presented in this chapter**, the **eigenvector** is a vector whose elements tell us what fraction of the system each class will have (well, the fraction that each class will *approach*) after a long time. This is also called the **long-term equilibrium (or steady) state** of the system.
 - (**) Recall that the matrices we are working with right now all have a special form- they are transfer matrices; we will soon see that this particular definition for eigenvector applies only to this special case and will need to be generalized

2. (8.2) Eigenvectors

Motivating Example

- Suppose we are modeling a non-fatal disease where you are either susceptible to acquiring the disease (S), you are infected with the disease (I), or you have recovered from the disease and are susceptible again (see Example 6.6)
- Suppose in this model, each day 10% of the susceptibles become infected, and 20% of the infected recover and become susceptible again
- Suppose that at time $t = 0$ we have 297 susceptible individuals and three infected individuals, and we assume that no new individuals enter the population and no one in the population dies or leaves. Thus, the population size remains at a constant size of 300 individuals

2. (8.2) Eigenvectors

Motivating Example

- Then, the transfer matrix T that models the daily change in this population is:

$$\mathbf{T} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

- And the vector describing the initial population with respect to this disease is:

$$\mathbf{x}(0) = \begin{bmatrix} 297 \\ 3 \end{bmatrix}$$

- What do we expect to happen over time? How many individuals do we expect to be infected after 50 days? 100 days? 365 days?

2. (8.2) Eigenvectors

Motivating Example

- Recall that we found a way to formulate and answer such questions fairly effortlessly:

$$\mathbf{x}(t) = \mathbf{T}^t \cdot \mathbf{x}(0)$$

- Here's what happens the first few days:

$$\mathbf{x}(1) = \mathbf{T} \cdot \mathbf{x}(0) = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 297 \\ 3 \end{bmatrix} = \begin{bmatrix} 268 \\ 32 \end{bmatrix}$$

$$\mathbf{x}(2) = \mathbf{T}^2 \cdot \mathbf{x}(0) = \begin{bmatrix} 0.83 & 0.34 \\ 0.17 & 0.66 \end{bmatrix} \cdot \begin{bmatrix} 297 \\ 3 \end{bmatrix} = \begin{bmatrix} 248 \\ 52 \end{bmatrix}$$

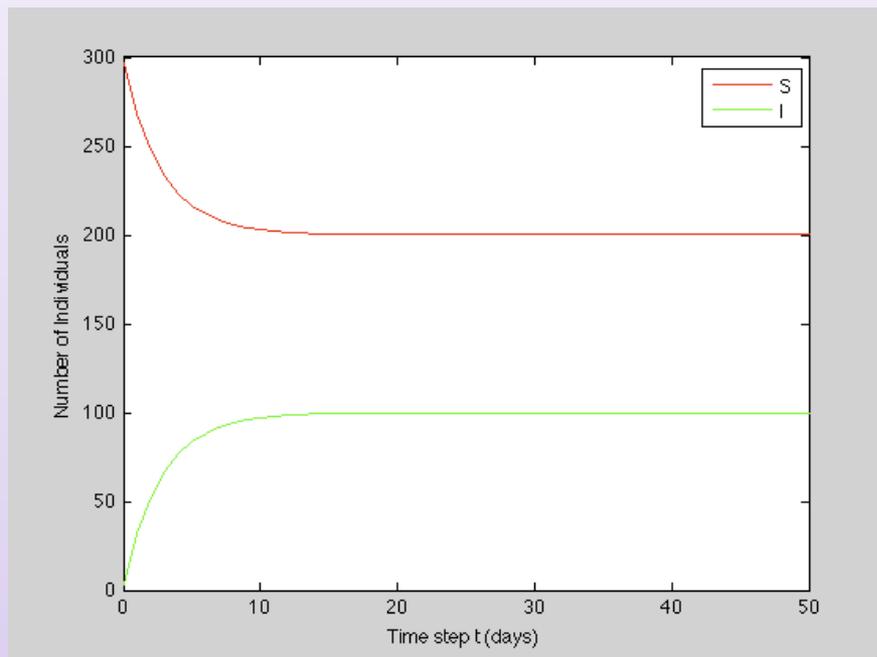
$$\mathbf{x}(3) = \mathbf{T}^3 \cdot \mathbf{x}(0) = \begin{bmatrix} 0.78 & 0.44 \\ 0.22 & 0.56 \end{bmatrix} \cdot \begin{bmatrix} 297 \\ 3 \end{bmatrix} = \begin{bmatrix} 233 \\ 67 \end{bmatrix}$$

We see that the number of susceptibles is decreasing, while the number of infecteds is increasing. Does this trend go on indefinitely?

2. (8.2) Eigenvectors

Motivating Example

- As before, we use MATLAB to make a plot showing the number of individuals in each class over the first 51 days:



- It definitely appears that the number of individuals in each class is no longer changing after about 20 days; that is, it appears that the system has reached an equilibrium

2. (8.2) Eigenvectors

Motivating Example

- But to go ahead and answer the question that was posed:

$$\mathbf{x}(50) = \mathbf{T}^{50} \cdot \mathbf{x}(0) = \begin{bmatrix} 0.67 & 0.67 \\ 0.33 & 0.33 \end{bmatrix} \cdot \begin{bmatrix} 297 \\ 3 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

$$\mathbf{x}(100) = \mathbf{T}^{100} \cdot \mathbf{x}(0) = \begin{bmatrix} 0.67 & 0.67 \\ 0.33 & 0.33 \end{bmatrix} \cdot \begin{bmatrix} 297 \\ 3 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

$$\mathbf{x}(365) = \mathbf{T}^{365} \cdot \mathbf{x}(0) = \begin{bmatrix} 0.67 & 0.67 \\ 0.33 & 0.33 \end{bmatrix} \cdot \begin{bmatrix} 297 \\ 3 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

- In fact,

$$\mathbf{x}(17) = \mathbf{T}^{17} \cdot \mathbf{x}(0) = \begin{bmatrix} 0.67 & 0.67 \\ 0.33 & 0.33 \end{bmatrix} \cdot \begin{bmatrix} 297 \\ 3 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

2. (8.2) Eigenvectors

Motivating Example

- So, the vector $[200 \ 100]^T$ is an eigenvector for the given transfer matrix.
- While the population structure is at equilibrium, this does **not** mean that individuals are no longer moving between the susceptible and infected classes
- There is still movement between the classes, however, the movement is such that the *number* in each class remains constant
- While looking at such a plot is helpful, we need to develop a *mathematical* method for determining this eigenvector
- Suppose, then, that we don't know the equilibrium distribution

2. (8.2) Eigenvectors

Motivating Example

- Moreover, assume that after t days, the system has reached an equilibrium state. This means that the distribution vector for $\mathbf{x}(t+1)$ will be the same as the one for $\mathbf{x}(t)$.

Mathematically, we have:

$$\mathbf{x}(t+1) = \mathbf{x}(t) \Leftrightarrow \mathbf{T} \cdot \mathbf{x}(t) = \mathbf{x}(t)$$

$$\Leftrightarrow \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 0.9x + 0.2y \\ 0.1x + 0.8y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

2. (8.2) Eigenvectors

Motivating Example

- This last equation implies that:

$$\begin{array}{l} 0.9x + 0.2y = x \\ 0.1x + 0.8y = y \end{array} \Leftrightarrow \begin{array}{l} 9x + 2y = 10x \\ x + 8y = 10y \end{array} \Leftrightarrow \begin{array}{l} x = 2y \\ x = 2y \end{array}$$

- The 2 equations are, in fact, equivalent and so we have only 1 equation in 2 unknowns
- We will see that this is typical when finding eigenvectors; that is, you will have a system of n equations (or less) in $n+1$ unknowns (this is an underdetermined system of algebraic equations)
- What does this mean for us as we attempt to keep working to find an eigenvector?

2. (8.2) Eigenvectors

Motivating Example

- Well, an equation like ours has many solutions. The equation $x=2y$ is solved by the following values for x and y :

x	2	4	2π	200	...
y	1	2	π	100	...

- This means that all of the following are eigenvectors of our matrix:
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2\pi \\ \pi \end{bmatrix}, \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$
- Though there are many different solutions, they *are* all related; namely, they all satisfy $x=2y$!

2. (8.2) Eigenvectors

Motivating Example

- So, even though they aren't unique, they are *unique up to a constant multiple*
- We can *normalize* them and thus obtain a unique *form* for an eigenvector
- To do this, simply add the components of the vector and then divide each component by that sum
- Let's normalize the 4 eigenvectors we found,

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2/(2+1) \\ 1/(2+1) \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \quad \text{All the same!} \quad \begin{bmatrix} 2\pi \\ \pi \end{bmatrix} \rightarrow \begin{bmatrix} 2\pi/3\pi \\ \pi/3\pi \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 4/(4+2) \\ 2/(4+2) \end{bmatrix} = \begin{bmatrix} 4/6 \\ 2/6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \quad \begin{bmatrix} 200 \\ 100 \end{bmatrix} \rightarrow \begin{bmatrix} 200/300 \\ 100/300 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

2. (8.2) Eigenvectors

Motivating Example

- So, the procedure to finish the problem:

$$x = 2y$$

- We choose *any* value for *either* variable and solve for the other. A convenient choice in this

case is to let $y=1$. Then $x=2$
and we have the eigenvector:

$$\longrightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Now we normalize:

$$\longrightarrow \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

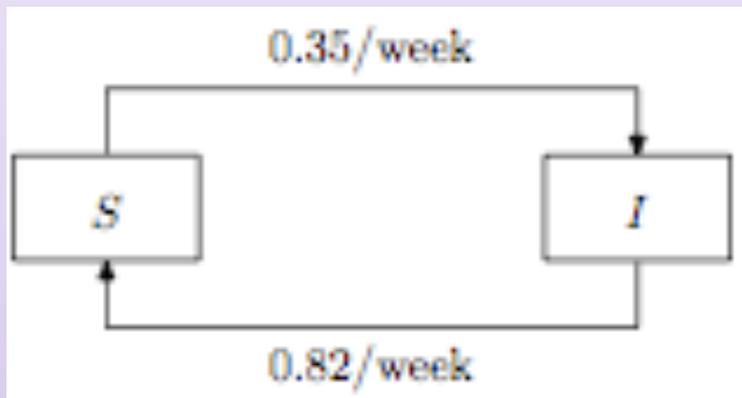
- And we scale so that we have 300 individuals:

$$\longrightarrow 300 \cdot \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

2. (8.2) Eigenvectors

Homework Exercise 8.8

- Suppose you are modeling a non-fatal infectious disease. You assume the people within the population you are modeling are either susceptible to infection, or infected. The following flow diagram shows the rates at which individuals flow from one category to the other:



$$\longrightarrow \mathbf{T} = \begin{bmatrix} 0.65 & 0.82 \\ 0.35 & 0.18 \end{bmatrix}$$

2. (8.2) Eigenvectors

Homework Exercise 8.8

- (a) Find the eigenvector that describes the system's equilibrium structure.

Solution: We seek a vector \mathbf{x} such that when \mathbf{T} acts on it (by multiplication) the resulting product is \mathbf{x} :

$$\mathbf{T}\mathbf{x} = \mathbf{x} \Leftrightarrow \begin{bmatrix} 0.65 & 0.82 \\ 0.35 & 0.18 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Leftrightarrow 0.65x + 0.82y = x$$

$$\Leftrightarrow 0.35x + 0.18y = y$$

$$\Leftrightarrow 100(0.35x + 0.18y) = 100(y)$$

$$\Leftrightarrow 35x + 18y = 100y$$

$$\Leftrightarrow x = \frac{82}{35}y$$

For a transfer matrix these 2 equations contain the same information (check this on your own); so we choose either one of them to find the eigenvector.

2. (8.2) Eigenvectors

Homework Exercise 8.8

- (a) Find the eigenvector that describes the system's equilibrium structure.

Solution: (cont'd) Choosing $y=35$ (I can choose any number I want; this particular choice is convenient), we obtain the eigenvector:

$$\begin{bmatrix} 82 \\ 35 \end{bmatrix} \xrightarrow{\text{Normalizing, we obtain:}} \begin{bmatrix} 82/(82 + 35) \\ 35/(82 + 35) \end{bmatrix} = \begin{bmatrix} 82/117 \\ 35/117 \end{bmatrix} \approx \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

2. (8.2) Eigenvectors

Homework Exercise 8.8

- (b) Suppose the rate of infection increases from 0.35/week to 0.50/week. How does this change the equilibrium structure?

Solution:

$$\mathbf{T}\mathbf{x} = \mathbf{x} \Leftrightarrow \begin{bmatrix} 0.50 & 0.82 \\ 0.50 & 0.18 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$0.5x + 0.82y = x$$

\Leftrightarrow

$$0.5x + 0.18y = y$$

$$\Leftrightarrow 50x + 18y = 100y$$

$$\Leftrightarrow x = \frac{82}{50}y$$

2. (8.2) Eigenvectors

Homework Exercise 8.8

- (b) Suppose the rate of infection increases from 0.35/week to 0.50/week. How does this change the equilibrium structure?

Solution: (cont'd) Choosing $\gamma=50$ we obtain the eigenvector:

$$\begin{bmatrix} 82 \\ 50 \end{bmatrix} \xrightarrow{\text{Normalizing, we obtain:}} \begin{bmatrix} 82/(82 + 50) \\ 50/(82 + 50) \end{bmatrix} = \begin{bmatrix} 82/132 \\ 50/132 \end{bmatrix} \approx \begin{bmatrix} 0.62 \\ 0.38 \end{bmatrix}$$

- As expected, the infection rate increase results in a larger proportion of infected individuals at the equilibrium state

2. (8.2) Eigenvectors

Homework Exercise 8.8

- (c) Suppose the rate of infection decreases from 0.35/week to 0.25/week. How does this change the equilibrium structure?

- Solution:

$$\mathbf{T}\mathbf{x} = \mathbf{x} \Leftrightarrow \begin{bmatrix} 0.75 & 0.82 \\ 0.25 & 0.18 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} & 0.75x + 0.82y = x \\ \Leftrightarrow & 0.25x + 0.18y = y \end{aligned}$$

$$\Leftrightarrow 25x + 18y = 100y$$

$$\Leftrightarrow x = \frac{82}{25}y$$

2. (8.2) Eigenvectors

Homework Exercise 8.8

- (c) Suppose the rate of infection decreases from 0.35/week to 0.25/week. How does this change the equilibrium structure?

Solution: (cont'd) Choosing $\gamma=25$ we obtain the eigenvector:

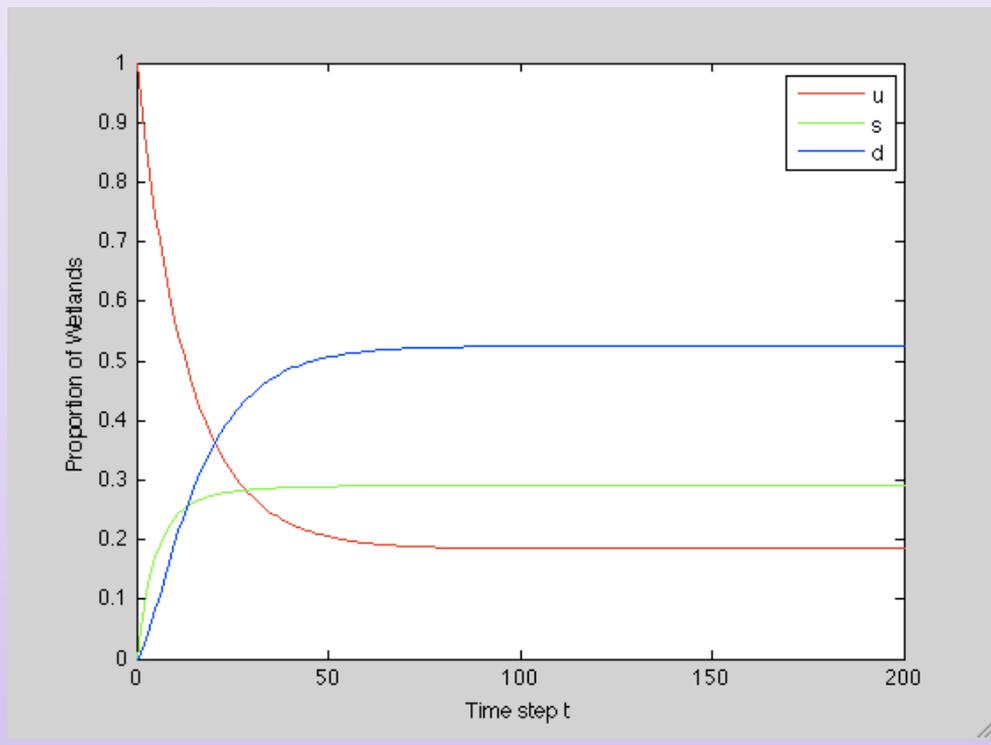
$$\begin{bmatrix} 82 \\ 25 \end{bmatrix} \xrightarrow{\text{Normalizing, we obtain:}} \begin{bmatrix} 82/(82 + 25) \\ 25/(82 + 25) \end{bmatrix} = \begin{bmatrix} 82/107 \\ 25/107 \end{bmatrix} \approx \begin{bmatrix} 0.77 \\ 0.23 \end{bmatrix}$$

- As expected, the infection rate decrease results in a larger proportion of susceptible individuals at the equilibrium state

2. (8.2) Eigenvectors

Example 7.5

- This is the plot from slide 3 above. It shows the dynamics for the ecological succession model we built in examples 6.5 and 7.5. We are now in a position to determine the long term equilibrium structure mathematically.



$$\mathbf{v}(100) = \begin{bmatrix} 0.1850 \\ 0.2907 \\ 0.5243 \end{bmatrix}$$

$$\mathbf{v}(200) = \begin{bmatrix} 0.1844 \\ 0.2908 \\ 0.5248 \end{bmatrix}$$

2. (8.2) Eigenvectors

Example 7.5

$$\mathbf{T}\mathbf{x} = \mathbf{x} \Leftrightarrow \begin{bmatrix} 0.94 & 0.02 & 0.01 \\ 0.05 & 0.86 & 0.06 \\ 0.01 & 0.12 & 0.93 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$.94x + .02y + .01z = x$$

$$\Leftrightarrow .05x + .86y + .06z = y$$

$$.01x + .12y + .93z = z$$

$$-6x + 2y + z = 0$$

$$\Leftrightarrow 5x - 14y + 6z = 0$$

$$x + 12y - 7z = 0$$

2. (8.2) Eigenvectors

Example 7.5

$$\begin{array}{rcl} -6x + 2y + z = 0 & & x - 0.3514z = 0 \\ 5x - 14y + 6z = 0 & \Rightarrow & y - 0.5541z = 0 \\ x + 12y - 7z = 0 & & 0 = 0 \end{array}$$

Choosing $z=1$, we obtain the eigenvector:

$$\begin{bmatrix} 0.3514 \\ 0.5541 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalizing, we obtain:}} \begin{bmatrix} 0.3514/1.9055 \\ 0.5541/1.9055 \\ 1/1.9055 \end{bmatrix} = \begin{bmatrix} 0.1844 \\ 0.2908 \\ 0.5248 \end{bmatrix}$$

This is what we expected; recall: $\mathbf{v}(100) = \begin{bmatrix} 0.1850 \\ 0.2907 \\ 0.5243 \end{bmatrix}$, $\mathbf{v}(200) = \begin{bmatrix} 0.1844 \\ 0.2908 \\ 0.5248 \end{bmatrix}$

3. (8.3) Stability

What about different initial conditions?

- Recall from the motivating example above that in finding an eigenvector, we never used the initial vector:

$$\mathbf{x}(0) = \begin{bmatrix} 297 \\ 3 \end{bmatrix}$$

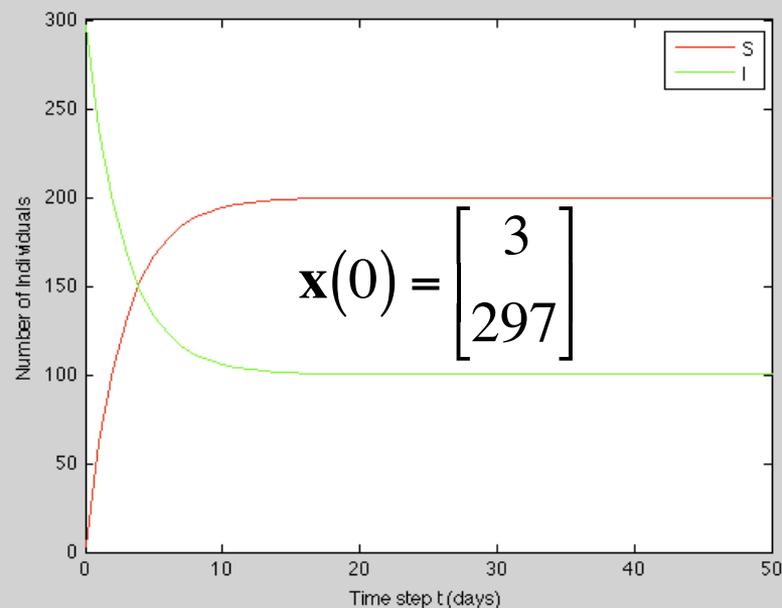
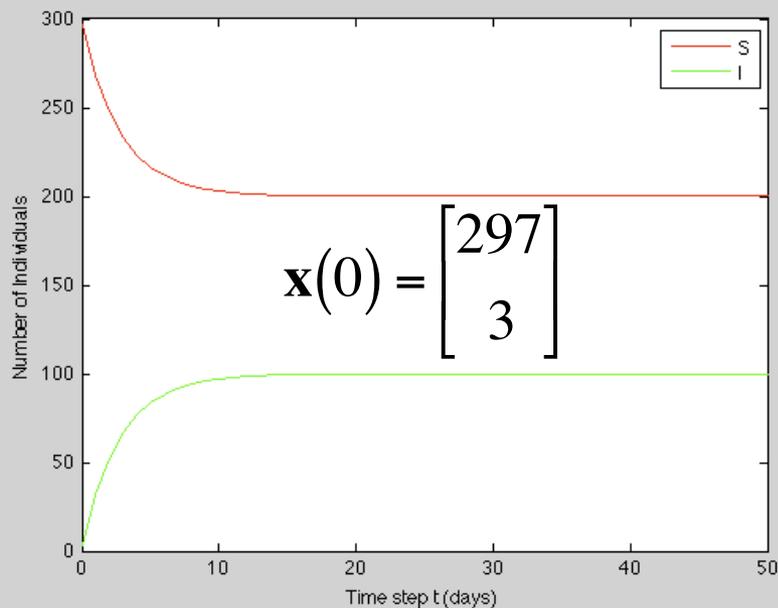
- In other words, the long-term population structure (for these models) *does not depend* on the initial condition
- For example, suppose we take the initial condition to be the other way around- initially 297 individuals are infected and only 3 are susceptible:

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 297 \end{bmatrix}$$

3. (8.3) Stability

What about different initial conditions?

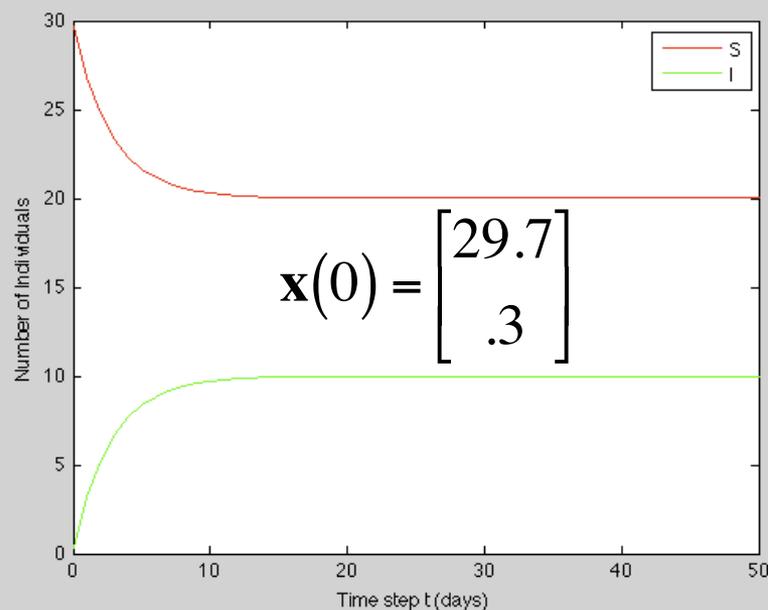
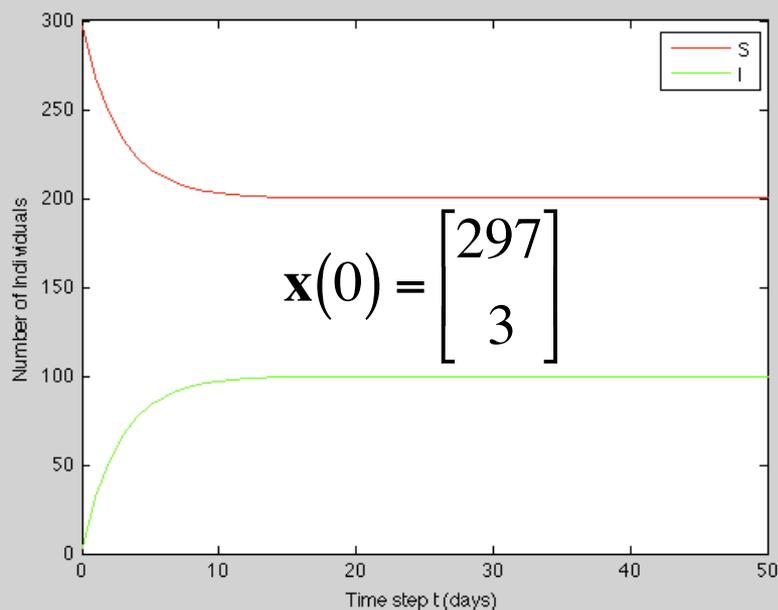
- Comparing the plots for each initial condition, we see that after (different) transient dynamics, the 2 scenarios have the same long-term equilibrium structure:



3. (8.3) Stability

What about different initial conditions?

- We may also consider initial conditions that differ in terms of the population size. Compare the plots for a population size of 300 individuals to one with 30 individuals:



- Again, the 2 scenarios have the same long-term equilibrium structure

Homework

- Chapter 8: 8.1-8.10